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# Limiting Load in Concrete Plates with Cracks: a Cell Method (CM)-Based Calculation

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## ABSTRACT

In this study, the problem of finding the limiting load in infinite plates with internal cracks is extended to the non-linear field. In particular, results concerning concrete plates in bi-axial loading are shown. The analysis is performed in discrete form, by means of the Cell Method. The discrete analysis allows us to identify the crack initiation without using the stress intensity factors. This simplifies the computation in cracked solids of finite dimensions. An example of computation in finite solids, the skew-symmetric four-point bending beam, is provided.

## INTRODUCTION

For finding the minimum load required to propagate a crack (limiting load), the variational principle of the most common crack theories has been used over the past three decades (Parton and Morozov 1978). Criteria for the initiation of crack propagation can be obtained on the basis

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of both energy and force considerations. Historically, at first an energy fracture criterion was proposed by A. A. Griffith in 1920 and G. R. Irwin formulated a force criterion in 1957, while the same time demonstrating the equivalence of the two criteria. The Irwin force criterion for crack extension and the equivalent Griffith energy criterion completely solve the question of the limiting equilibrium state of a cracked continuous elastic body. Nevertheless, there exists a number of other formulations also establishing the limiting equilibrium state of a cracked body. Among these, the best known models are those of Leonov and Panasyuk (1959), Dugdale (1960), Wells (1961), Novozhilov (1969), and McClintock (1958).

Usually, the variational problem of finding the limiting load is reduced to that of finding extreme points of a function of several variables (Parton and Morozov 1978).

In the present paper, the variational approach has been abandoned in favour of a discrete formulation of the crack propagation problem. The numerical calculation is then performed by a new numerical method for solving field equations: the Cell Method (CM) (Tonti in press).

The numerical code for crack initiation analysis with the CM has been developed by E. Ferretti (Ferretti in press). In this study, the code has been extended to provide results in the case of a concrete plate (Fig. 1) tensioned at infinity by a load of intensity $p_x = kp_0$ parallel to the x-axis, and $p_y = p_0$ parallel to the y-axis. The plate has an initial straight crack of length $2l_0$ oriented at an angle $a_0$ to the x-axis. The minimum load required to propagate the crack from the ends of the cut is provided for various values of $k$, $a_0$.

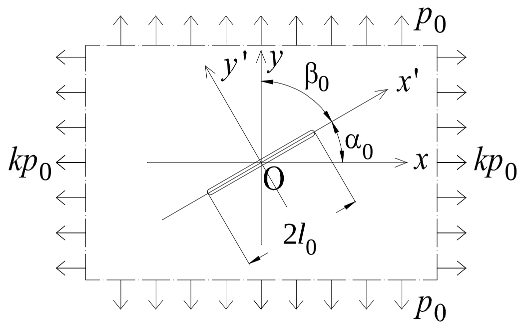


Fig. 1. Load and geometrical set-up of the cracked infinite plate.

CRACK EXTESION CRITERION

The crack limiting load can be determined using a variety of criteria:

- the maximal normal stress criterion;
- the maximal strain criterion;
- the minimum strain energy density fracture criterion;
- the maximal strain energy release rate criterion;
- the damage law criteria.

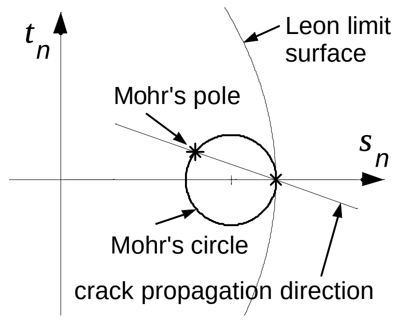


Fig. 2. Leon limit surface in the $s_n - t_n$ plane.

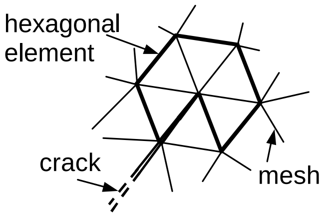


Fig. 3. Hexagonal element for analysis in the $s_n - t_n$ plane.

In the present paper, the crack extension condition is studied in the Mohr-Coulomb plane. The limiting load is computed as the load satisfying the condition of tangency between the Mohr's circle and the Leon limit surface (Fig. 2). Said $c$ the cohesion, $f_c$ the compressive strength, $f_{tb}$ the tensile strength, the Leon criterion is expressed as:

$$t_n^2 = \frac{c}{f_c}\left(\frac{f_{tb}}{f_c} + s_n\right). \quad (1)$$

To identify the Mohr's circle for the tip neighbourhood, an hexagonal element was inserted at the tip (Ferretti in press, Fig. 3), as to establish a correspondence between the tip stress field and the attitudes. The propagation direction is then derived as the direction of the line joining the tangent point to the Mohr's pole (Fig. 2).

NUMERICAL RESULTS

Numerical results concerning the infinite concrete plate loaded as shown in Fig. 1 are here presented. For symbols and conventions, refer to the same Fig. 1. The concrete constitutive law adopted in this study is monotonically nondecreasing, in accordance with the identification procedure provided in Ferretti 2001 (Fig. 4).

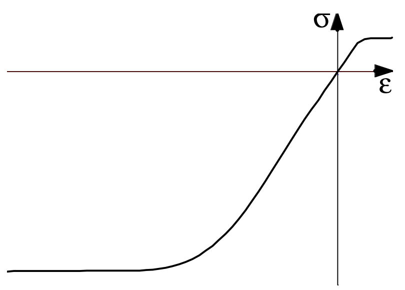


Fig. 4. Adopted constitutive law for concrete in mono-axial load.

The $p_y$ analysis for $k = 0$ and $a_0 = 45°$, plotted in greyscale on the deformed configuration of a finite area around the crack, is shown in Fig. 5.a. In this figure, the darker colour corresponds to the maximal tensile stress, while the lighter colour corresponds to $p_y = 0$. In Fig. 5.b, the $p_y$ analysis is performed in 3D, with the level lines plotted in the plane $p_y = 0$. The boxed area in the plane $p_y = 0$ is the area of the figure Fig. 5.a. As all the level lines are internal to the boxed area, it can be assumed with good approximation that this area represents the stress extinction zone.

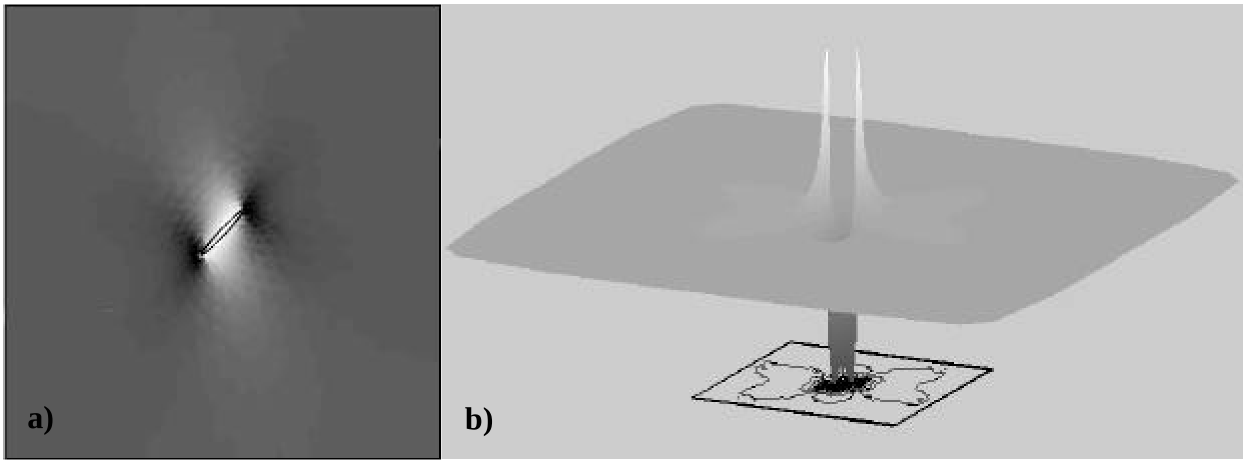


Fig. 5. a) Greyscale plane $p_y$ analysis; b) 3D $p_y$ analysis and lines of equal $p_y$.

The normalised limiting load in the direction of the y-axis, $p_y \big/ \max_{a_0,k} p_y$, is plotted in Fig. 6.a for each value of the angle $a_0$ ($b_0 = 90 - a_0$) and for the factor *k* equal, respectively, to 0, $1/4$, $1/2$, $3/4$ and 1. This figure exhibits a $p_y$ limiting load increasing with $a_0$ in the field $0 \le k < 1$, stating that $a_0 = 0$ is the critical crack orientation angle for all the biaxial load conditions with the x-component lower than the y-component. The constant behaviour of the line obtained for $k = 1$ is in good agreement with the homogeneous state of stress represented by this load condition. In this case, all the crack orientations to the x-axis return the same limiting load.

The load conditions corresponding to a given *k* and to its reciprocal, $1/k$, are symmetric with respect to a crack inclination $a_0 = 45°$. In the intent to plot limiting load curves for *k* and $1/k$ satisfying the symmetry with respect to $a_0 = 45°$, the ideal limiting load $p_{id}$ was defined as:

$$p_{id} = \sqrt{p_x^2 + p_y^2} = p_0\sqrt{k^2 + 1} \,. \tag{2}$$

The limiting load curves in the $a_0 - p_{id} \big/ \max_{a_0,k} p_{id}$ plane for $0 \le k \le +\infty$ are plotted in figure Fig. 6.b. Each line in Fig. 6.b represents the function $f_k = p_{id}(a_0)$, returning the value of $p_{id}$ at a given value of *k*. All the $f_k$ lines together represent the function $f = p_{id}(a_0, k)$, defining the value of the ideal limiting load in function of the crack orientation and the ratio between the loads in direction of the x- and y-axis. The lower envelope of the $f_k$ curves was plotted in thick line in figure Fig. 6.b. This line represents the function *F* defined as:

$$F = F\left(a_0, k\right) = \min_{a_0=\text{cost}} p_{id}\left(a_0, k\right).$$

The projection on the $a_0$-$k$ plane of the $F$ function, said $\overline{F}$, returns the value of $k^{cr}$ (which is the same, the couple of normalised values $p_x^{cr}$ and $p_y^{cr}$), corresponding to the minimum value of ideal limiting load $p_{id}$ for a given $a_0$. In other words, the function $\overline{F}$ can be expressed as the solution of the following differential problem:

$$\overline{F} = k^{cr}\left(a_0\right) : \frac{\partial p_{id}\left(a_0, k\right)}{\partial k} = 0.$$

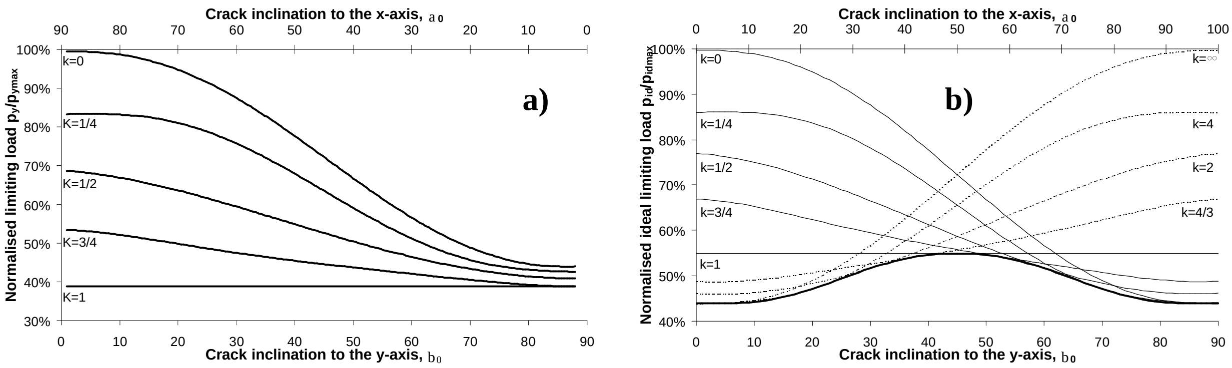


Fig. 6. a) Normalised limiting load $p_y$ in function of the crack inclination $a_0/b_0$ for $0 \le k \le 1$; b) Normalised ideal limiting load $p_{id}$ in function of the crack inclination $a_0/b_0$ for $0 \le k \le +\infty$.

The third coordinate $y$ was defined as:

$$y = 1 - e^{-k}. \tag{3}$$

With this position, it is possible to give a finite 3D representation of $p_{id}$, in function of $a_0$ ($b_0$) and $k$. Some examples of the 3D surfaces obtained for different values of axis orientation are given in Fig. 7. The Fig. 7.a is the 3D equivalent representation of the Fig. 6.b, while the Fig. 7.f is obtained by a 90° rotation of the normalised $p_{id}$ surface around the normalised $p_{id}$-axis. This last plot represents the function $g = p_{id}\left(k, a_0\right)$ in the $k$ first independent variable and $a_0$ second independent variable. The meridians of $g$ give the functions $g_{a_0} = p_{id}\left(k\right)$ for a given value of $a_0$. The lower envelope of the $g_{a_0}$ functions gives the function:

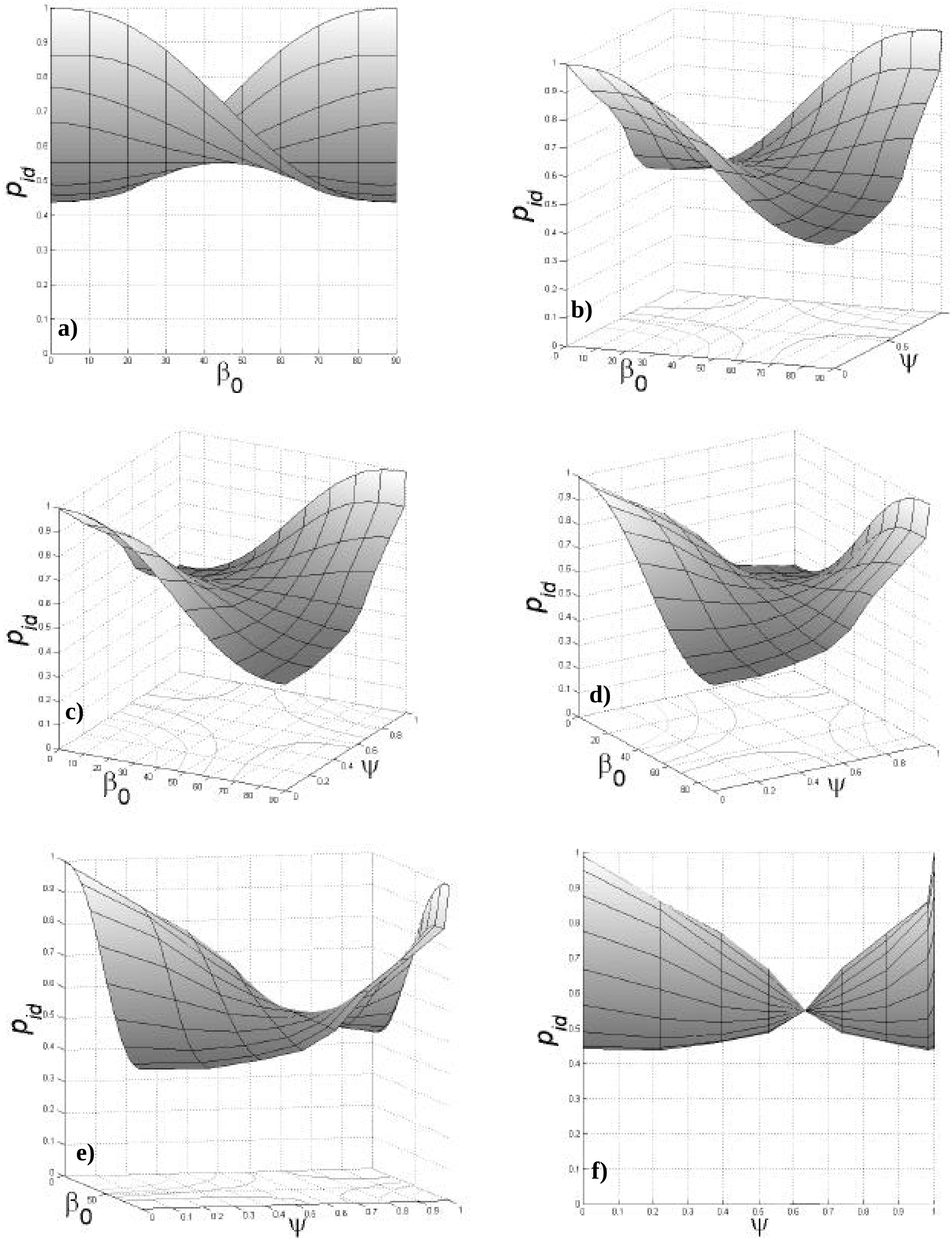


Fig. 7. Some examples of finite 3D representation of $p_{id}$, in function of $b_0$ and *k*.

$$G = G(k, \alpha_0) = \min_{k=\text{cost}} p_{id}(k, \alpha_0).$$

Said $\overline{G}$ the projection on the $k$-$\alpha_0$ plane of the $G$ function, $\overline{G}$ returns the angle $\alpha_0^{cr}$ minimising $p_{id}$ at a given $k$. $\overline{G}$ is the solution of the following differential problem:

$$\overline{G} = \alpha_0^{cr}(k) : \frac{\partial p_{id}(\alpha_0, k)}{\partial \alpha_0} = 0.$$

As all the $g_{\alpha_0}$ functions intersect for $k = 1$ and they do not have any other point in common, the $\alpha_0 = \alpha_0^{cr}$ minimising $p_{id}$ at a given $k$ results equal to:

$$\alpha_0^{cr} = \begin{cases} 0 & 0 \le k < 1 \\ [0, 90] & k = 1 \\ 90 & 1 < k \le +\infty \end{cases}$$

Finally, by way of illustration of the code potentialities, results regarding the computation on a geometry of finite dimensions are here presented. The computation on finite geometries allow to avoid the quantitative discrepancy between the experimental and via- stress intensity factors calculated results (Parton and Morozof 1978), due to the effect of the specimen boundaries on the stress field around the growing crack. In the case of a beam with two opposite pre-cracks submitted to skew-symmetric mixed-mode load in four-points, the numerical result was so accurate as to allow the complete description of the crack path (Fig. 8). In Fig. 8, the darker colour corresponds to the maximal compressive stress and the lighter colour corresponds to the maximal tensile stress.

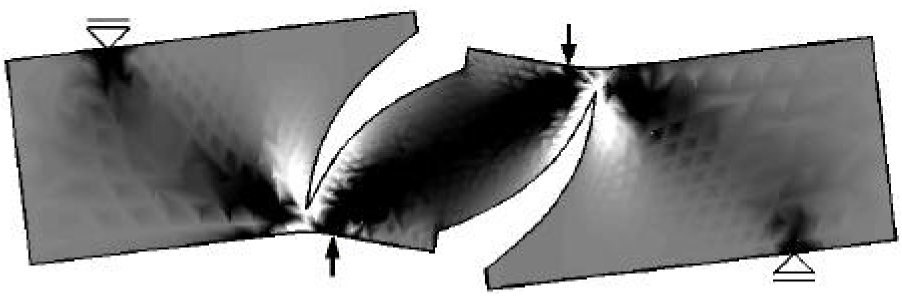

Fig. 8. Displacement - stress analysis of the beam under skew-symmetric four point bending.

## CONCLUSIONS

A first study on tensioned concrete plates was presented, based on an innovative size-insensitive constitutive law.

The adopted numerical model, founded on the CM, allows analysis in the discrete. The crack initiation is then studied without using the stress intensity factors. It was shown how the numerical results for plates of infinite dimensions loaded in Mode I are satisfying with respect to the load and geometrical parameters. It was also shown how the results on a solid of finite dimensions are highly accurate. Moreover, it is remarkable how the analysis for finite solids is performed directly, without having to apply corrective factors to the solution on an infinite geometry in the same load conditions.

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