The Cell Method as a Case of Bialgebra

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Abstract: - The Cell Method (CM) associates any physical variable with the geometrical and topological features, usually neglected by the differential formulation. This goal is achieved by abandoning the habit to discretize the differential equations. The governing equations are then derived in algebraic manner directly, by means of the global variables. In the original formulation of the CM, the association between physical variables and geometry is made on the basis of physical considerations. In this paper, we analyze the same association under the mathematical point of view. This allows us to view the CM as a geometric algebra, which is an enrichment of the exterior algebra. The p-space elements and their inner and outer orientations are derived inductively. They are obtained from the outer product of the geometric algebra and the features of p-vectors. Space and time global variables are treated in a unified four-dimensional space/time cell complex, whose elementary cell is the tesseract. Moreover, the configuration variables with their topological equations, on the one hand, and the source variables with their topological equations, on the other hand, are viewed as a bialgebra and its dual algebra.

Key-Words: - Cell Method, Geometric Algebra, Inner Orientation, Outer Orientation, Bialgebra, Tesseract.

1 Introduction

Of special importance for the philosophy of the Cell Method (CM) are the geometric interpretations of the operations on vectors, provided by both the exterior and geometric algebra, and the notions of extension of a vector by another vector, multivector (or p-vector), dual vector space, bialgebra, and covector. The geometric approach allows us to view the space elements and the time elements as p-vectors of a geometric algebra, all inductively generated by the outer product of the geometric algebra. From the attitude and orientation of p-vectors, we then derive the two kinds of orientation for p-vectors, inner and outer orientations, which apply to both the space and the time elements. We also discuss how the orientation of a p-vector is induced by the orientation of the $(p-1)$-vectors on its boundary, and how the inner orientation of the attitude vector of a vector equals the outer orientation of its covector. This establishes an isomorphism between the orthogonal complement and the dual vector space of any subset of vectors. One of the most remarkable consequences of the relationship between inner and outer orientations is that the outer orientation depends on the dimension of the embedding space, while the inner orientation does not.

2 Some Basics of the Exterior Algebra

The exterior algebra provides an algebraic setting in which to answer geometric questions. It is the largest algebra that supports an alternating product on vectors. Its product is the exterior product, or wedge product.

The exterior product of any number k of vectors can be defined and is sometimes called a k-blade. It lives in a geometrical space known as the k-th exterior power. The magnitude of the resulting k-blade is the volume of the k-dimensional parallelotope whose sides are the given vectors, just as the magnitude of the scalar triple product of vectors in three dimensions gives the volume of the parallelepiped spanned by those vectors. In particular, The exterior product of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \wedge \mathbf{b}$, is called a 2-vector, or bivector, and lives in a space called the exterior square, a geometrical vector space that differs from the original space of vectors. The magnitude of $\mathbf{a} \wedge \mathbf{b}$ can be interpreted as the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, which, in three dimensions, can also be computed using the cross product of the two vectors.

Since the exterior product is antisymmetric, $\mathbf{b} \wedge \mathbf{a}$ is the negation of the bivector $\mathbf{a} \wedge \mathbf{b}$, producing the opposite orientation.

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Any vector space, V, has a corresponding dual vector space (or just dual space), V^*. Given any vector space V over a field F, the algebraic dual space V^*, also called the ordinary dual space, or simply the dual space, is defined as the set of all linear maps (linear functionals) from V to F:

$$\varphi : V \to F \text{ , } v \mapsto \varphi(v) \text{ .} \qquad (1)$$

If V is finite-dimensional, then V^* has the same dimension as V. Dual vector spaces for finite-dimensional vector spaces can be used for studying tensors.

The pairing of a functional φ in the dual space V^* and an element x of V is sometimes denoted by a bracket:

$$\varphi(x) = [\varphi, x] = \langle \varphi, x \rangle \text{ .} \qquad (2)$$

The pairing defines a non-degenerate bilinear mapping:

$$[\bullet,\bullet] : V^* \times V \to F \text{ .} \qquad (3)$$

Specifically, every non-degenerate bilinear form on a finite-dimensional vector space V gives rise to an isomorphism from V to V^*, $\langle \bullet,\bullet \rangle$. Then, there is a natural isomorphism:

$$V \to V^*,\ v \mapsto v^* ; \qquad (4)$$

given by:

$$v^*(w) := \langle v, w \rangle ; \qquad (5)$$

where $v^* \in V^*$ is said to be the dual vector of $v \in V$.

A topology on the dual space, X^*, of a topological vector space, X, over a topological field, $\mathbf{K}$, can be defined as the coarsest topology (the topology with the fewest open sets) such that the dual pairing $X^* \times X \to \mathbf{K}$ is continuous. This turns the dual space into a locally convex topological vector space. This topology is called the weak* topology, that is, a weak topology defined on the dual space X^*. In order to distinguish the weak topology from the original topology on X, the original topology is often called the strong topology. If X is equipped with the weak topology, then addition and scalar multiplication remain continuous operations, and X is a locally convex topological vector space.

Elements of the algebraic dual space V^* are sometimes called covectors, or 1-forms, and are denoted by bold, lowercase Greek. They are linear maps from V to its field of scalars.

if V is a vector space of any (finite) dimension, then the level sets of a linear functional in V^* are parallel hyperplanes in V, and the action of a linear functional on a vector can be visualized in terms of these hyperplanes, or p-planes, in the sense that the number of (1-form) hyperplanes intersected by a vector equals the interior product between the covector and the vector (Fig.1).

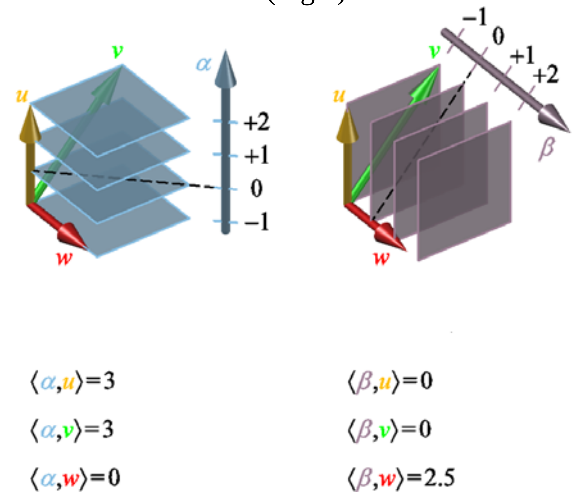

Fig.1: Linear functionals (1-forms) α, β, and vectors u, v, w, in 3d Euclidean space.

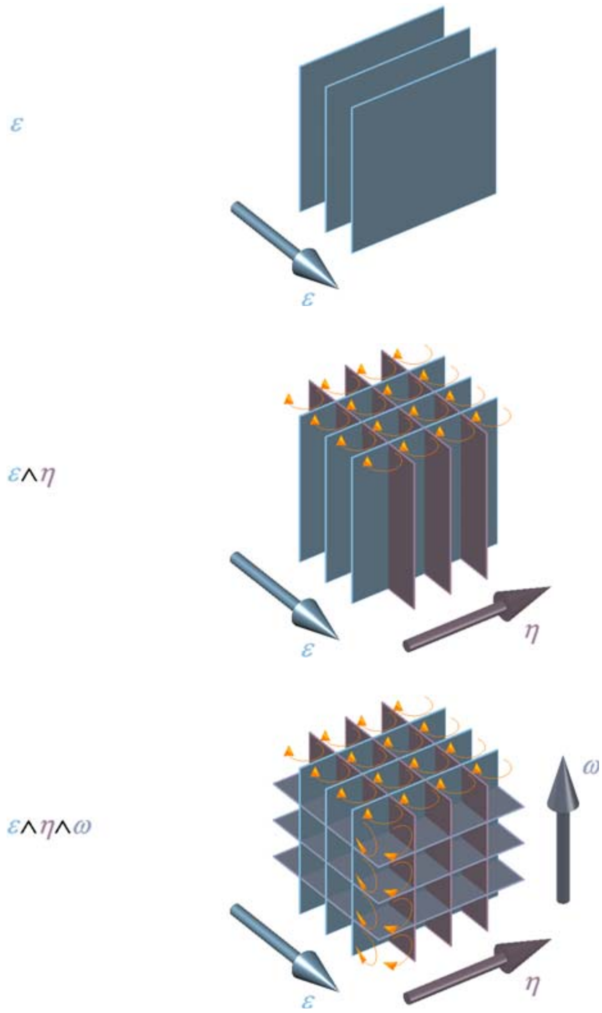

Fig.2: Geometric interpretation for the exterior product of k 1-forms (ε, η, ω) to obtain an k-form ("mesh" of coordinate surfaces, here planes), for $k = 1,2,3$. The "circulations" show orientation.

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In multilinear algebra, a multilinear form, or k-form, is a map of the type:

$$f : V^k \to \mathbf{K}; \tag{6}$$

where V is a vector space over the field $\mathbf{K}$, which is separately linear in each its k variables. The k-forms are generated by the exterior product on covectors (Fig.2).

3 An Insight into Geometric Algebra

Vector algebra and geometric algebra (GA) are alternative approaches to providing additional algebraic structures on vector spaces, with geometric interpretations. Vector algebra is specific to Euclidean three-space, while geometric algebra uses multilinear algebra and applies in all dimensions and signatures. They are mathematically equivalent in three dimensions, though the approaches differ.

Geometric algebra gives emphasis on geometric interpretations and physical applications. A geometric algebra is the Clifford algebra $\mathcal{C}\ell(V,Q)$ of a vector space over the field of real numbers endowed with a quadratic form.

The distinguishing multiplication operation that defines the geometric algebra as a unital ring is the geometric product. Taking the geometric product among vectors can yield bivectors, trivectors, or general p-vectors. The addition operation combines these into general multivectors. This includes, among other possibilities, a well-defined sum of a scalar and a vector, an operation that is impossible by the traditional vector addition.

We may write the geometric product of any two vectors a and b as the sum of a symmetric product and an antisymmetric product:

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba). \tag{7}$$

The symmetric product in Eq. (7) defines the inner product of vectors a and b:

$$a \cdot b := \frac{1}{2}(ab + ba) = \frac{1}{2}\left((a+b)^2 - a^2 - b^2\right); \tag{8}$$

which is a real number, because it is a sum of squares, and is not required to be positive definite. It is not specifically the inner product on a normed vector space.

The antisymmetric product in Eq. (7) is equal to the exterior product of the contained exterior algebra and defines the outer product of vectors a and b:

$$a \wedge b := \frac{1}{2}(ab - ba). \tag{9}$$

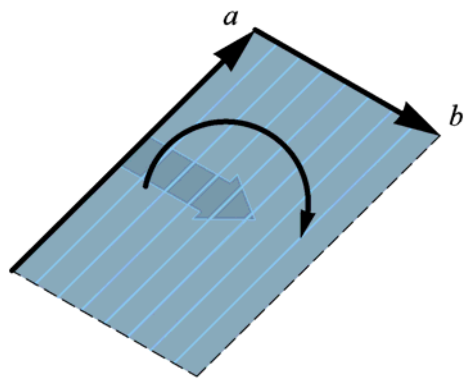

Fig.3: The extension of vector a along vector b provides the geometric interpretation of $a \wedge b$.

Geometrically, the outer product $a \wedge b$ can be viewed by placing the tail of the arrow b at the head of the arrow a and extending vector a along vector b (Fig.3). The resulting entity is a two-dimensional subspace, and we call it a bivector. It has an area equal to the size of the parallelogram spanned by a and b. The senses of a and b orientate the sides of the parallelogram and define a sense of traversal of its boundary. In the case of Fig.3, the traversal sense is a clockwise sense, which can be depicted by a clockwise arc.

The geometric interpretation of the outer product $b \wedge a$ is achieved by placing the tail of the arrow a at the head of the arrow b and extending vector b along vector a. This reverses the circulation of the boundary, while it does not change the area of the parallelogram spanned by a and b.

In conclusion, the geometric product in Eq. (7) can be written as the sum between a scalar and a bivector:

$$ab = a \cdot b + a \wedge b. \tag{10}$$

The scalar and the bivector are added by keeping the two entities separated, in the same way in which, in complex numbers, we keep the real and imaginary parts separated.

One can consider the Clifford algebra $\mathcal{C}\ell(V,Q)$ as an enrichment (or more precisely, a quantization) of the exterior algebra $\Lambda(V)$ on V with a multiplication that depends on Q. For nonzero Q there exists a canonical linear isomorphism between $\Lambda(V)$ and $\mathcal{C}\ell(V,Q)$, whenever the ground field K does not have characteristic two. That is, they are naturally isomorphic as vector spaces, but with different multiplications.

The p-vectors are charged with three attributes, or features: attitude, orientation, and magnitude. The second feature, taken singularly and combined with the first feature, gives rise to the two kinds of orientation in space, inner and outer orientations.

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3.1 Inner Orientation of Space Elements

The second feature of p-vectors, the orientation, is, more properly, an inner orientation, because it does not depend on the embedding space. The term "inner" refers to the fact that the circulations are defined for the boundaries of the elements, by choosing an order for the vertexes. Therefore, we move and stay on the boundaries of the elements, without going out from the elements themselves.

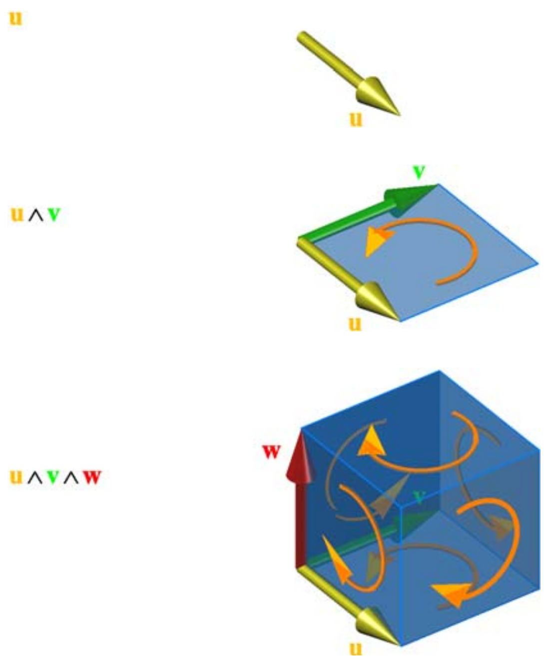

Fig.4: Geometric interpretation for the exterior product of p vectors to obtain an p-vector, where $p = 1,2,3$. The "circulations" show the inner orientation.

In GA, the inner orientation is the geometric interpretation of the exterior geometric product among vectors. In particular, the inner orientation of a plane surface can be viewed as the orientation of the exterior product between two vectors $\mathbf{u}$ and $\mathbf{v}$ (the bivector $\mathbf{u} \wedge \mathbf{v}$) of the plane on which the surface lies (Fig.4). Analogously, the inner orientation of a volume can be viewed as the orientation of the exterior product between three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$) of the three-dimensional space containing the volume (Fig.4).Fig.4

It is worth noting that the inner orientation of a surface is not positive or negative in itself. Neither choosing the sign of the inner orientation can be considered an arbitrary convention. Providing the inner orientation of a surface with a sign makes sense only when the surface is "watched" by an external observer, that is, only when the surface is studied in an embedding space of dimension greater than 2, the dimension of the surface.

The six faces of the positive trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ in Fig.4 have a negative inner orientation when they are watched by an external observer, while they have a positive inner orientation when they are watched by a local observer that is inside the volume. This happens since the inner volume of the trivector is the intersection of the six positive half-spaces, that is, the half-spaces of the six observers that watch the positive surfaces originated by the trivector. By relating the sign of the inner orientation to the external observer also in this second case, the positive inner orientation of a volume is the one watched by the external observer. As a consequence, the inner orientation of a volume is positive when the inner orientations of all its faces are negative, as in Fig.4.

The concept of inner orientation defined above did not apply to zero-dimensional vector spaces (points). However, since it is useful to be able to assign different inner orientations to a point, in this paper we extend the outer product to zero-dimensional vectors:

$$\mathbf{P} \wedge \mathbf{Q} \triangleq \mathbf{u}; \tag{11}$$

which has the geometrical meaning of point $\mathbf{P}$ extended toward point $\mathbf{Q}$. The extension of the outer product preserves the antisymmetric property of the product, since $\mathbf{Q} \wedge \mathbf{P}$ (point $\mathbf{Q}$ extended toward point $\mathbf{P}$) is the negation of $\mathbf{P} \wedge \mathbf{Q}$:

$$\mathbf{Q} \wedge \mathbf{P} = -\mathbf{u}. \tag{12}$$

Analogously, a point extended by a vector results in an oriented length, which can be represented by the vector itself (Fig.5). Consequently, a bound vector with origin in $\mathbf{P}$, $(\mathbf{P}, \mathbf{u})$, can be seen as the outer product between $\mathbf{P}$ and the free vector $\mathbf{u}$:

$$\mathbf{P} \wedge \mathbf{u} \triangleq (\mathbf{P}, \mathbf{u}). \tag{13}$$

Then, since the bound vector $(\mathbf{P}, \mathbf{u})$ is often denoted by simply $\mathbf{u}$, as its free vector, we can also write

$$\mathbf{P} \wedge \mathbf{u} = \mathbf{u}. \tag{14}$$

For consistency, we must therefore define the outer product between the vector $\mathbf{u}$ and the point $\mathbf{P}$ as the negation of $\mathbf{u}$:

$$\mathbf{u} \wedge \mathbf{P} \triangleq -\mathbf{u}. \tag{15}$$

In analogy to the direction of the vector product $\mathbf{u} \times \mathbf{v}$, which is orthogonal both to $\mathbf{u}$ and $\mathbf{v}$, the result of the operation $\mathbf{P} \wedge \mathbf{Q}$, defined in $\mathbb{R}^1$ on elements of $\mathbb{R}^0$, has the direction of a line that is orthogonal both to $\mathbf{P}$ and $\mathbf{Q}$. In three-dimensional space, where we can define infinite sub-spaces of dimension 1, each provided with its own basis, this operation produces elements in the direction of any

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line of the three-dimensional space. Being orthogonal to each direction of the three-dimensional space, the point is orthogonal to the three-dimensional space itself and to each volume of the space.

We can define two inner orientations of a point, the outward and the inward orientations (Fig.6). In the first case, the point is called a source, while, in the second case, is called a sink.

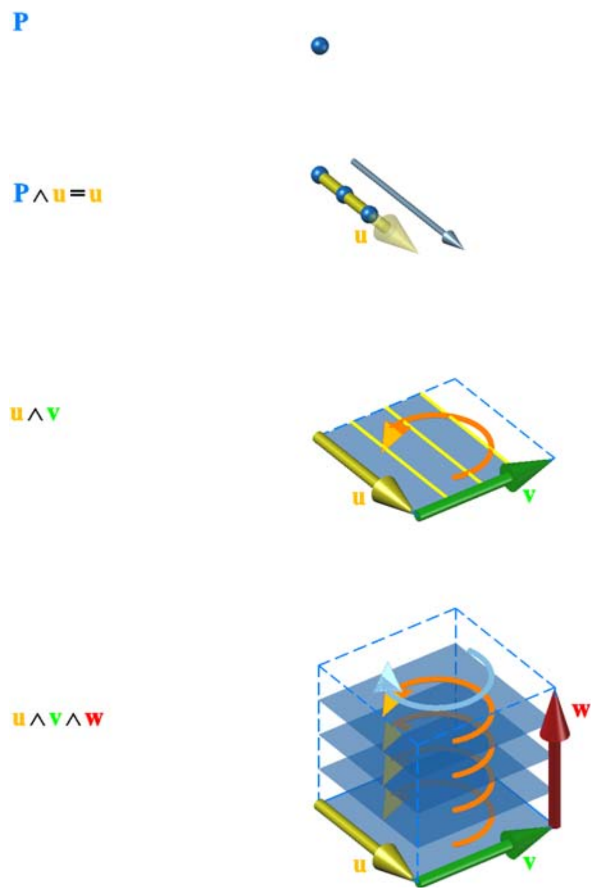

Fig.5: The inner orientation of a p-space element is induced by the $(p-1)$-space elements on its boundary.

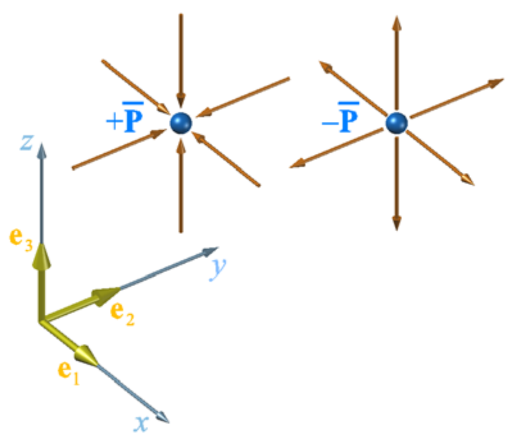

Fig.6: Positive and negative **i**nner orientations of a point.

By making use of the notion of observer in this latter case too, each incoming line can be viewed as the sense along which the external observer watches the point. In this sense, a sink is a point with a positive inner orientation, while a source is a point with a negative inner orientation (Fig.6).

Even the trivector can be viewed as an extension. In fact, it is originated by a bivector extended by a third vector (Fig.5).

In conclusion, since the positive or negative inner orientation of a p-space element is induced by the positive or negative inner orientation of the $(p-1)$-space elements on its boundary, we derive the inner orientations and their signs inductively (Fig.5). This allows us to extend the procedure for finding the inner orientation of the space elements to spaces of any dimension.

3.2 Outer Orientation of Space Elements

The attitude is part of the description of how p-vectors are placed in the space they are in. Thus, the notion of attitude is related to the notion of embedding of a p-vector in its space, or space immersion. In particular, a vector in three dimensions has an attitude given by the family of straight lines parallel to it (possibly specified by an unoriented ring around the vector), a bivector in three dimensions has an attitude given by the family of planes associated with it (possibly specified by one of the normal lines common to these planes), and a trivector in three dimensions has an attitude that depends on the arbitrary choice of which ordered bases are positively oriented and which are negatively oriented.

Between a p-vector and its attitude there exists the same kind of relationship that exists between an element a of a set X and the equivalence class of a in the quotient set of X by a given equivalence relationship. In the special case of the attitude of a vector in the three-dimensional space, the set is that of the straight lines and the equivalence relationship is that of parallelism between lines. One of the invariants of the equivalence relation of parallelism is the family of planes that are normal to the lines in a given equivalence class. Since we can choose any of the parallel planes for representing the invariant, we can speak both in terms of family of parallel planes and in terms of one single plane.

Similar considerations may also be applied to the relationship between bivectors and their attitudes, or trivectors and their attitudes. Thus, we can describe the attitude of a p-vector either in terms of its

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equivalence class (the family of parallel lines, when the p-vector is a vector), or in terms of its class invariant (the family of normal planes, when the p-vector is a vector), that is, the equivalence class of its orthogonal complement. In particular, the attitude of a vector $\mathbf{u}$ can be viewed as a family of normal planes (Fig.7), each one originated by the translation of a plane normal to $\mathbf{u}$, along the direction of $\mathbf{u}$ (the planes span the direction of $\mathbf{u}$). Equivalently, the attitude of $\mathbf{u}$ can be represented by an arbitrary plane of the family of normal planes.

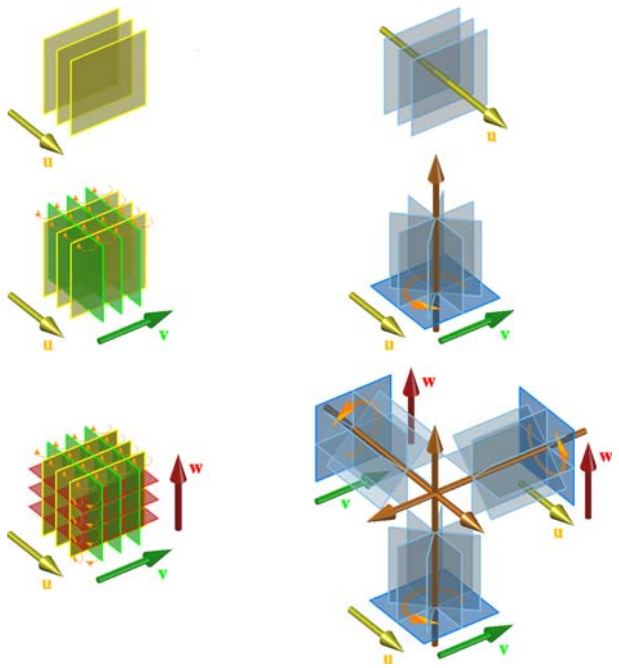

Fig.7: Geometric interpretation of the attitude of a p-vector in terms of class invariants.

Analogously, the attitude of a bivector $\mathbf{u} \wedge \mathbf{v}$ can be viewed as two families of parallel planes (Fig.7), the first family normal to $\mathbf{u}$ and the second family normal to $\mathbf{v}$ (the planes span both the directions of $\mathbf{u}$ and $\mathbf{v}$). Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent in their common plane, the planes that span both the directions of $\mathbf{u}$ and $\mathbf{v}$ originate all the planes normal to $\mathbf{u} \wedge \mathbf{v}$, that is, all the planes parallel to $\mathbf{u} \times \mathbf{v}$. These planes can be represented by the line of intersection between an arbitrary plane of the first family and an arbitrary plane of the second family (Fig.7). The intersection line is parallel to all planes of the two families and to vector $\mathbf{u} \times \mathbf{v}$.

Finally, the attitude of a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ can be viewed as three families of parallel planes (Fig.7), provided that the three families are normal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, respectively (the planes span the three directions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$). If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, then the three families originate all the plane of the three-dimensional space. A possible representation of all the planes of the three-dimensional space, under the equivalence relation of parallelism, is achieved by choosing a point of the space and considering the set of all the planes that contain the point. Being common to all the planes, the point can be used for representing the whole set of planes, which, in turn, represents all the planes of the three-dimensional space.

In conclusion, as for the inner orientation of the p-vectors, also the attitude of the p-vectors is defined inductively, starting from the 1-vector. This allows us to define the attitude of the p-vectors even in dimension greater than 3.

The same family of parallel planes represents both the set of planes that are normal to $\mathbf{u}$ and the set of hyperplanes of $\mathbf{u}^*$, the dual vector of $\mathbf{u}$. Consequently, the attitude of the class invariant of a vector $\mathbf{u}$ equals the attitude of the covector $\mathbf{u}^*$. This is ultimately a consequence of the Riesz representation theorem, which allows us to represent a covector by its related vector. This establishes a bijective correspondence between the attitude of the orthogonal complement of a vector $\mathbf{u}$ and the attitude of the covector, $\mathbf{u}^*$, of $\mathbf{u}$.

The bijective correspondence extends also to the second feature, that is, the orientations of a vector and its covector, since the order of the hyperplanes is determined by the sense of $\mathbf{u}$. This allows us to define a second type of orientation for the covector $\mathbf{u}^*$, which we call the outer orientation since it is induced by the (inner) orientation of $\mathbf{u}$ and has the geometrical meaning of sense of traversal of the hyperplanes of $\mathbf{u}^*$. In doing so, we have established a bijective correspondence between the inner orientation of a vector and the outer orientation of its covector. On the other hand, since it is always possible to define an inner orientation for $\mathbf{u}^*$ (by choosing a basis bivector for $\mathbf{u}^*$), the duality between vectors and covectors will result in an outer orientation for $\mathbf{u}$, induced by the inner orientation of $\mathbf{u}^*$. Therefore, the inner orientation of a covector induces an outer orientation on its vector.

Moreover, since the equivalence classes of $\mathbf{u}^*$ are in bijective correspondence with the attitude of $\mathbf{u}$, to fix the inner orientation of $\mathbf{u}^*$ is also equivalent to fixing an orientation, which is an inner orientation, for the attitude of $\mathbf{u}$. In doing so, the attitude of $\mathbf{u}$ becomes an attitude vector, and its inner orientation equals the outer orientation of $\mathbf{u}$. Therefore, by providing the attitude with an orientation, we establish an isomorphism between the orthogonal complement and the dual vector

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space of any subset of vectors. This means that the pairing between the geometric algebra and its dual can be described by the invariants of the equivalence relation of parallelism.

In conclusion, we can define the orientation of a vector by providing either its inner orientation or the inner orientation of its attitude vector (which is also the outer orientation of the vector). The latter, in turn, is equal to the inner orientation of the covector.

The relationship between the inner and outer orientations and the related notion of orthogonal complement (or dual element) are implicit, both in mathematics and physics. They are given by the right-hand rule, which is equivalent to the right-hand grip rule and the right-handed screw rule. We make them explicit in this paper because they are at the basis of the CM description of physics.

The dual of a p-dimensional space element has dimension $n-p$, in the n-dimensional space. This means that the outer orientation depends on the dimension of the embedding space, while the inner orientation does not.

4 The Cell Method

We can classify the physical variables according to their nature, global or local. Broadly speaking, the global variables are those variables that are neither densities nor rates of other variables. The field variables are obtained from the global variables as densities of space global variables and rates of time global variables. Due to their point-wise nature, they are local variables.

In the differential formulation, some variables arise directly as functions of points and time instants, while the remaining variables are reduced to points and time instants functions by performing densities and rates. Thus, the physical variables of the differential formulation are point-wise and/or instant-wise field functions.

In the algebraic formulation of the CM, on the contrary, we use global variables. In doing so, the algebraic formulation preserves the length and time scales of the global physical variables. Therefore, the physical variables, in spatial description, turn out to be naturally associated with one of the four space elements (point, line, surface, and volume, which are denoted by $\mathbf{P}$, $\mathbf{L}$, $\mathbf{S}$, and $\mathbf{V}$) and/or with one of the two time elements (time instant and time interval, which are denoted by $\mathbf{I}$ and $\mathbf{T}$).

Global variables are essential to the philosophy of the CM, since, by using these variables, it is possible to obtain an algebraic formulation directly and, what is most important, the global variables involved in obtaining the formulation do not have to be differentiable functions. The main difference between the two formulations – algebraic and differential – lies precisely in the fact that the limit process is used in the latter. In effect, since calculating the densities and rates of the domain variables is based on the assumption that global variables are continuous and differentiable, the range of applicability of differential formulation is restricted to regions without material discontinuities or concentrated sources, while that of the algebraic formulation is not restricted to such regions.

In the CM, the global variables are associated with the related space elements of the cell complexes, that is, the four space elements, $\mathbf{P}$, $\mathbf{L}$, $\mathbf{S}$, and $\mathbf{V}$, of the cell complexes. This allows us to describe global variables directly. The inner and outer orientations of the space elements used by the CM, assumed on the basis of physical considerations until now, find their mathematical foundations in the previous discussion on the GA. Also the generalization of the time axis is made by means of cell complexes, which, in this second case, are associated with the two time elements, that is, time instant, $\mathbf{I}$, and time intervals, $\mathbf{T}$.

4.1 Inner and Outer Orientations of Time Elements

Finding the orientations of the time elements could be viewed in the same way that finding the inner and outer orientations of points and lines in a one-dimensional space. In fact, the time axis defines a one-dimensional cell complex, where the time instants, $\mathbf{I}$, are points (nodes) and the time intervals, $\mathbf{T}$, are the line segments that connect the points (the time instants are the boundary, or the faces, of the time intervals). Moreover, in a one-dimensional space the dual (orthogonal complement) of a point is a line segment and the dual of a line segment is a point. As far as the inner orientation is concerned, all the time instants, both those along the positive semi-axis and those along the negative semi-axis, are sinks. Thus, they have an inward inner orientation (Fig.8). Finally, we can decide that the inner orientation of the time intervals is the same as the orientation of the time axis.

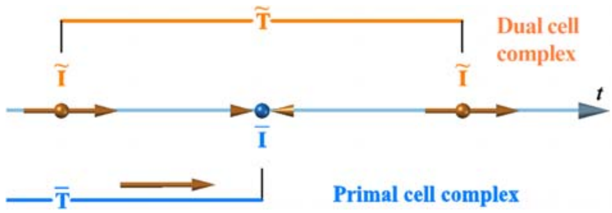

Fig.8: Time elements and their duals.

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After a more detailed analysis, however, it is clear that building a cell complex in time makes no sense in itself. In fact, in physics time has not importance in itself. It is just a variable, useful for describing how a physical phenomenon evolves. Now, since any physical phenomenon occurs in space, it follows that the time axis must always be related to one or more axes in space. The perception itself of the time is linked to bodies. Therefore, a cell complex in time must be two-dimensional, at least. In this paper, we propose to add a time axis to a three-dimensional cell complex where the cell of maximum dimension has been originated by a trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$. This gives rise to a four-dimensional space/time cell complex, whose cell of maximum dimension is a tesseract.

In multilinear algebra, a tesseract is a further element of the (graded) exterior algebra on a vector space. It is the four-dimensional analog of the cube, in the sense that it is to the cube as the cube is to the square. Each edge of a tesseract is of the same length and there are three cubes folded together around every edge (Fig.9). Just as the surface of the cube consists of six square faces, the hypersurface of the tesseract consists of eight cubical cells.

Note that the elements of a CM space/time 4-vector (a tesseract) are of different nature, since some p-cells are associated with a variation of the space variables, some other p-cells are associated with a variation of the time variables, and some other p-cells are associated with a variation of both the space and time variables. In particular, the points are associated with a variation of both the space and time variables. Therefore, we can say that there exists just one kind of points. As far as the others p-cells are concerned, on the contrary, we can define two different kinds of cells for each $p=1,2,3$. We will denote

- 1-cells of the kind “space”: the 1-cells that connect points associated with the same time instant, that is, the edges of the trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$ at a given instant (Fig.9);
- 1-cells of the kind “time”: the 1-cells that connect points associated with two adjacent time instants, that is, the time intervals (Fig.9);
- 2-cells of the kind “space”: the 2-cells that connect edges associated with the same time instant, that is, the faces of the trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$ at a given instant (Fig.10);
- 2-cells of the kind “space/time”: the 2-cells that connect edges associated with two adjacent time instants. The area of one of these faces is given by the product between a time interval and an edge of the trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$ (Fig.10);
- 3-cells of the kind “space”: the 3-cells that connect faces associated with the same time instant, that is, the volume of the trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$ at a given instant (Fig.11);
- 3-cells of the kind “space/time”: the 3-cells that are enclosed within faces associated with two adjacent time instants. The volume of one of these 3-cells is given by the product between a time interval and two edges of the trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$ (Fig.11).

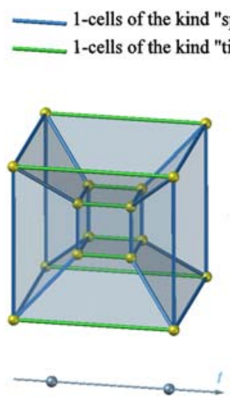

Fig.9: Different kinds of 1-cells in a space/time tesseract.

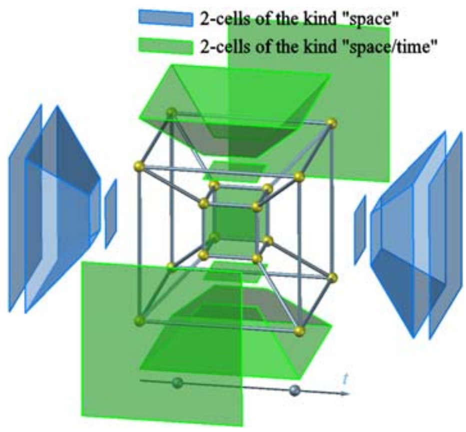

Fig.10: Different kinds of 2-cells in a space/time tesseract.

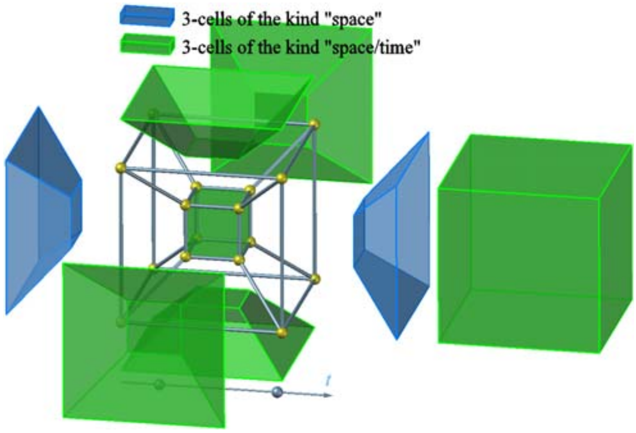

Fig.11: Different kinds of 3-cells in a space/time tesseract.

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In order to comply with the natural time sequence, from previous time instants to subsequent time instants, we will associate the elements of the left cube with the previous instant and the elements of the right cube with the subsequent instant. Consequently, the eight edges connecting the left to the right cube turn out to have an inner orientation from left to right, that is, the same orientation of the time axis (Fig.12). Moreover, since the same point of a four-dimensional space denotes both a point in space and a point in time (a time instant), it follows that the time instants have an inward inner orientation, that is, they are sinks.

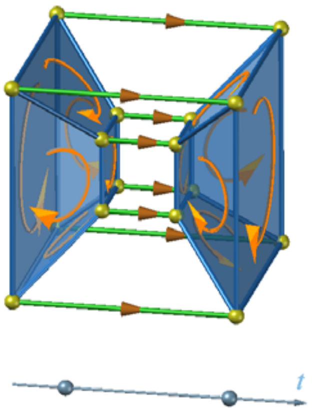

Fig.12: The CM tesseract: inner orientations on the 3-cells of the kind space and the 1-cells of the kind time.

When we associate the global space and time variables with the oriented elements of a 4-vector, we obtain the algebraic version of a four-dimensional Minkowski continuum, called spacetime, whose metric treats the time dimension differently from the three spatial dimensions. Spacetime is thus not a Euclidean space.

4.2 The Mathematical Structure of the CM

A further criterion for classifying the physical variables is based on the role they play in a theory. According to this second criterion, all physical variables belong to one of the following three classes:

- Configuration variables, describing the field configuration;
- Source variables, describing the field sources;
- Energetic variables, resulting from the multiplication of a configuration variable by a source variable.

The equations used to relate the configuration variables of the same physical theory to each other and the source variables of the same physical theory to each other are known as topological equations.

We have seen that, by providing the elements of a vector space with an inner orientation, the elements of the dual vector space turn out to be automatically provided with an outer orientation, as a consequence of the Riesz representation theorem. Now, due to the geometrical interpretation of the elements of the vector spaces, given by the geometric algebra, we can associate the elements of the two vector spaces with the geometrical elements of two cell complexes, where the elements of the second cell complex are the orthogonal complements of the corresponding elements in the first cell complex. Due to this association, by providing the elements of the first cell complex with an inner (or an outer) orientation, we induce an outer (or an inner) orientation on the second cell complex. Moreover, since the source variables requires an outer orientation, a proper description of a given physical phenomenon requires to use two cell complexes in relation of duality, not just one, as usually was done in computational physics before the introduction of the CM. In fact, it is true that the inner orientation of the elements of a vector space also induces an outer orientation on the elements of the same vector space and this may allow us to think that a single cell complex would be sufficient. Nevertheless, the association between the two orientations of the same cell complex is not automatic. There are always two possible criteria for establishing the correspondence between the two orientations, which depend on the orientation of the embedding space. Conversely, the relationship between inner (or outer) orientation of a cell complex and outer (or inner) orientation of its dual cell complex is derived from the Riesz representation theorem and does not depend on the orientation of the embedding space. Therefore, choosing to use two cell complexes, the one the dual of the other, instead of one single cell complex, is motivated by the need to provide a description of vector spaces that is independent of the orientation of the embedding space.

We will call the first complexes in space and time the primal cell complexes, or primal complexes, and the second complexes in space and time the dual cell complexes, or dual complexes.

The cell complexes are generalizations of the oriented graphs. Therefore, all the properties of the dual graphs naturally extend to the dual cell complexes. In particular, the dual graphs depend on a particular embedding. Since even the orthogonal complements (that is, the isomorphic dual vectors) and the outer orientation depend on the embedding, we will associate the outer orientation with the dual

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cell complex and will retain the inner orientation for the primal cell complex.

The most natural way for building the two cell complexes is starting from a primal cell complex made of simplices and providing this first cell complex with an arbitrary inner orientation. The set of the dual elements can then be chosen as any arbitrary set of staggered elements whose outer orientations provide the (known) inner orientations of the primal p-cells. In this sense, we can say that the outer orientations of the dual p-cells are induced by the inner orientations of the primal p-cells.

By associating the configuration variables with the primal p-cells, the set of topological equations between global configuration variables defines a geometric algebra on the space of global configuration variables, provided with a geometric product. The operators of these topological equations are generated by the outer product of the geometric algebra, which is equal to the exterior product of the enclosed exterior algebra. The dual algebra of the enclosed exterior algebra is the space of global source variables, associated with the dual p-cells, and is provided with a dual product that is compatible with the exterior product of the exterior algebra. The topological equations between global source variables arise from the adjoint operators of the primal operators. Finally, the pairing between the exterior algebra and its dual gives rise to the energetic variables, by the interior product. Since the reversible constitutive relations may be written in terms of energetic variables, because energy is the potential of the reversible constitutive relations, the reversible constitutive relations realize the pairing between the exterior algebra and its dual.

4 Conclusion

The configuration and the source variables used by the Cell Method have two different orientations, inner rather than outer orientation. This, together with the need to provide a description of vector spaces that is independent of the orientation of the embedding space, require to use two different cell complexes, whose geometrical elements stand by each other in relation of duality. The two cell complexes are called the primal and the dual cell complex. The elements of the first cell complex in space and first cell complex in time are associated with the variables endowed with an inner orientation, while the elements of the second cell complex in space and second cell complex in time are associated with the variables endowed with an outer orientation. As a consequence, the source variables are associated with the elements of the dual complex, and the configuration variables are associated with the elements of the primal complex.

In this paper we have shown how the configuration variables with their topological equations, on the one hand, and the source variables with their topological equations, on the other hand, define two vector spaces that are a bialgebra and its dual algebra. The operators of these topological equations are generated by the outer product of the geometric algebra, for the primal vector space, and by the dual product of the dual algebra, for the dual vector space. The topological equations in the primal cell complex are coboundary processes on even exterior discrete p-forms, while the topological equations in the dual cell complex are coboundary processes on odd exterior discrete p-forms. Being expressed by coboundary processes in two different vector spaces, compatibility and equilibrium can be enforced at the same time, with compatibility enforced on the primal cell complex and equilibrium enforced on the dual cell complex.

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