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The Cell Method: an Enriched Description of Physics Starting from the Algebraic Formulation

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Abstract: In several recent papers studying the Cell Method (CM), which is a numerical method based on a truly algebraic formulation, it has been shown that numerical modeling in physics can be achieved even without starting from differential equations, by using a direct algebraic formulation. In the present paper, our focus will be above all on highlighting some of the theoretical features of this algebraic formulation to show that the CM is not simply a new numerical method among many others, but a powerful numerical instrument that can be used to avoid spurious solutions in computational physics.

Keywords: Algebraic Formulation, Differential Formulation, Cell Method, Spurious Solutions, Nonlocality.

1 Introduction

From the onset of differential calculus, over three centuries ago [Newton (1687)], we have become accustomed to providing a differential formulation to each experimental law. Infinitesimal analysis has without doubt played a major role in the mathematical treatment of physics in the past, and will continue to do so in the future, but we must also be aware that, in using it, several important aspects of the phenomenon being described, such as its geometrical and topological features [Tonti (in press)], remain hidden. Moreover, applying the limit process introduces some limitations as regularity conditions must be imposed on the field variables. These regularity conditions, in particular those concerning differentiability, are the price we pay for using a formalism that is both very advanced and easy to manipulate.

Since the arrival of computers, differential equations have been discretized using one of various discretization methods (the finite element method FEM, the bound-

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ary element method BEM, the finite volume method FVM, the finite difference method FDM, etc.), since the numerical solution, which is no longer an exact solution, cannot be achieved for the most general case if a system of algebraic physical laws is not provided. Nevertheless, the very need to discretize the differential equations, in order to achieve a numerical solution, gives rise to the question of whether or not it is possible to formulate physical laws in an algebraic manner directly, through a direct algebraic formulation. We will see, in this paper, that this is possible and that a truly algebraic numerical method, the Cell Method (CM [Tonti (2001)]), besides being physically more appealing, also allows us to avoid some of the typical numerical problems of differential formulation, since it would appear that many numerical problems are purely the result of the generally consolidated custom of formulating the problem in differential form.

2 Basics of algebraic formulation

The starting point in algebraic formulation [Tonti (1998); Tonti (2001); Freschi, Giaccone, and Repetto (2008); Alotto, Freschi, and Repetto (2010); Alotto, Freschi, Repetto, and Rosso (2013)] is that only a few physical variables arise directly as functions of points and instants. Most of them are obtained by performing densities and rates on variables related to extended space elements and time intervals.

We will call:

- global variable in space, a variable that is not the line, surface or volume density of another variable;
- global variable in time, a variable that is not the rate of another variable.

2.1 Global variables and their features

Global variables are essential to the philosophy of the Cell Method, since, by using these variables, it is possible to obtain an algebraic formulation directly and, what is most important, the global variables involved in obtaining the formulation do not have to be differentiable functions. Therefore, by using the limit process on the mean densities and rates of the global variables, we can obtain the traditional field functions of the differential formulation.

The main difference between the two formulations – algebraic and differential – lies precisely in the fact that the limit process is used in the latter. In effect, since calculating the densities and rates of the domain variables is based on the assumption that global variables are continuous and differentiable, the range of applicability of differential formulation is restricted to regions without material discontinuities or concentrated sources, while that of the algebraic formulation is not restricted

to such regions [Ferretti (2004a,b); Ferretti, Casadio, and Di Leo (2008); Ferretti (2013)].

Among the possible classifications of physical variables, the one we will adopt in this work makes the distinction between:

- configuration variables, which describe the field configuration;
- source variables, which describe the field sources.

Displacements in solid mechanics, velocity in fluid dynamics, electric potential in electrostatics and temperature in thermal conduction are all examples of configuration variables, while forces in solid mechanics and fluid dynamics, masses in geodesy, electric charges in electrostatics, electric currents in magnetostatics and heat in thermal conduction are instead examples of source variables.

The equations used to relate the configuration variables of the same physical theory to each other and the source variables of the same physical theory to each other are known as topological equations, while those that relate configuration to source variables, of the same physical theory, are known as constitutive equations.

Since each physical phenomenon occurs in space, and space has a multi-dimensional geometrical structure, the physical variables themselves have a multi-dimensional geometrical content. As a consequence, each of the global physical variables is associated with one of the four space elements: point (**P**), line (**L**), surface (**S**) or volume (**V**). The association between physical variables and space elements in dimensions 0, 1, 2 and 3 is ignored in differential formulation, while it is emphasized in the Cell Method, where it becomes the corner stone for building the relative governing equations. As we will see better later, the fact that the Cell Method takes account of the association between physical variables and space elements is also the main reason why it does not present the spurious solutions of differential formulation [Pimprikar, Teresa, Roy, Vasu, and Rajan (2013); Qian, Han, and Atluri (2013); Kakuda, Nagashima, Hayashi, Obara, Toyotani, Katsurada, Higuchi, and Matsuda (2012); Kakuda, Obara, Toyotani, Meguro, and Furuichi (2012); Cai, Tian, and Atluri (2011); Chang (2011); Dong and Atluri (2011); Liu (2012); Jarak and Sori\'c, (2011); Liu and Atluri (2011); Liu, Dai, and Atluri (2011); Wu and Chang (2011); Cai, Paik, and Atluri (2010); Liu, Hong, and Atluri, (2010); Soares, D. Jr. (2010); Yeih, Liu, Kuo, and Atluri (2010); Zhu, Cai, Paik, and Atluri (2010)].

We will use algebraic topology notations to describe the four space elements **P**, **L**, **S** and **V**. In algebraic topology, it is usual to consider cell-complexes, and to denote the vertexes as 0-cells, the edges as 1-cells, the surfaces as 2-cells and the volumes as 3-cells (Fig. 1).

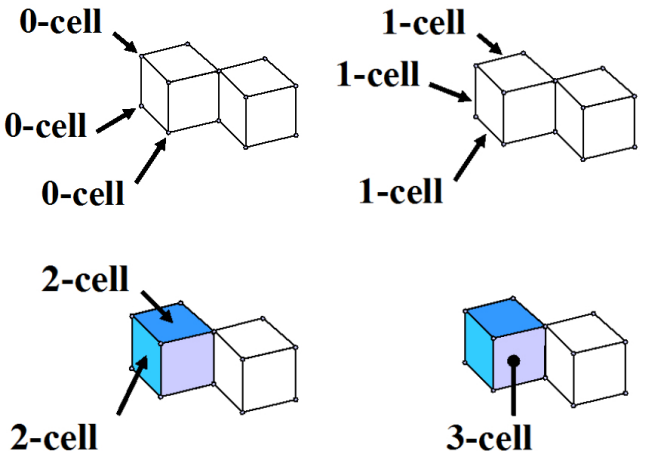

Figure 1: The four space elements in algebraic topology

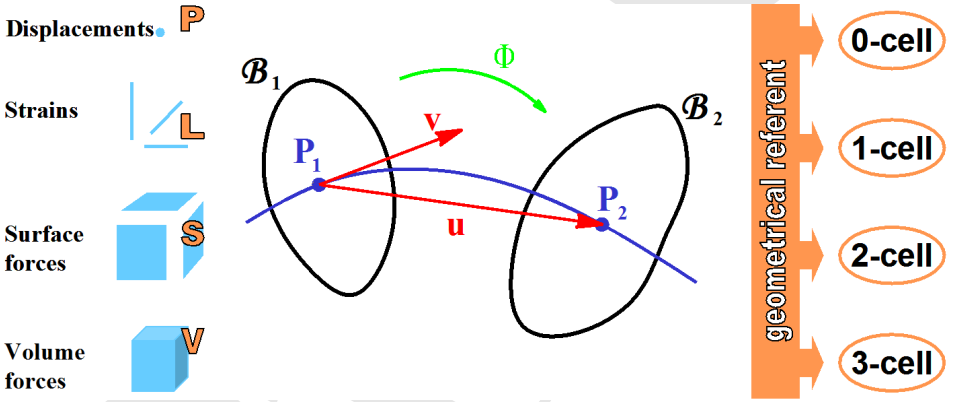

Figure 2: Association between space elements and variables in continuum mechanics

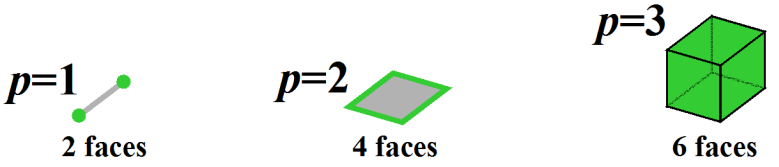

Figure 3: Faces of a p-cell of degree 1, 2 and 3.

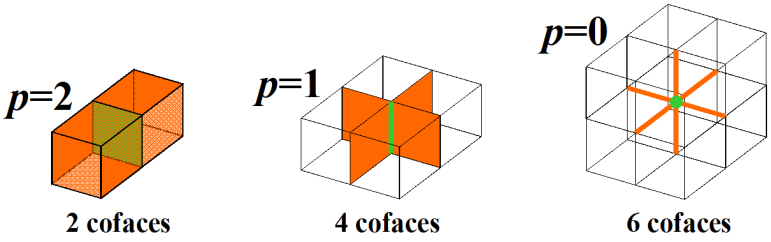

Figure 4: Cofaces of a p-cell of degree 0, 1 and 2.

In continuum mechanics, volume forces, which are source variables, are associated with 3-cells, since their geometrical referents are volumes (Fig. 2). Analogously, for surface forces, which are source variables, the geometrical referents (the surfaces) are 2-cells; for strains, which are configuration variables, the geometrical referents (the lines) are 1-cells; and for displacements, which are configuration variables, the geometrical referents (the points) are 0-cells.

Algebraic topology also involves the notions of face and coface. If we consider a cell-complex made of p-cells of degree 0, 1, 2 and 3, the $(p-1)$-cells that bound a given p-cell are the faces of the p-cell (Fig. 3). The set of faces of a p-cell defines the boundary of the p-cell.

The $(p+1)$-cells that have a given p-cell as a common face are the cofaces of that p-cell (Fig. 4). The set of cofaces of a p-cell defines the coboundary of the p-cell.

2.2 How the Cell Method works

The Cell Method has often been compared to the Direct or Physical Approach, initially used in the Finite Element Method [Huebner (1975); Livesley (1983); Fenner (1996)], or to the Finite Volume Method and the Finite Difference Method. In particular, the Cell Method may seem very similar to the vertex-based scheme of the FVM [Mavripilis (1995)]. However, on deeper analysis of the similarities and differences between the CM and other discrete methods, the CM is shown to be based on a new philosophy, where, for the moment, the CM is seen as the only truly algebraic method. In effect, the key point to bear in mind in building a truly algebraic formulation is that all operators must be discrete and use of the limit process must be avoided at each level of the formulation. The direct or physical approach is not suited to this, since it starts from the point-wise conservation equations of differential formulation (Fig. 5) and, for differential formulation, there is the need for field functions, which depend on the point position and the instant value. If the field functions are not described directly in terms of point position and instant values,

they can be obtained by calculating the densities and rates of the global variables, which are domain variables and depend on the point position and the instant value, and also on line extensions, areas, volumes and time intervals.

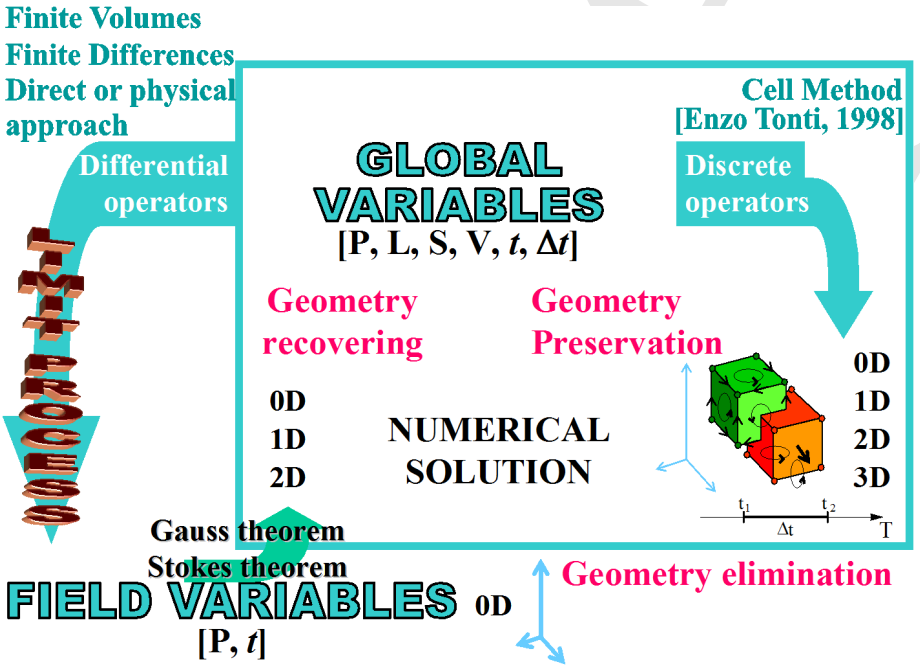

Figure 5: Building an algebraic formulation through the Direct or Physical Approach, the Finite Volume Method, the Finite Difference Method and the Cell Method

The space distribution of the point-wise field functions requires the introduction of coordinate systems (Fig. 5), whose purpose is to create a correspondence between the points of the space and the numbers, that is, their coordinates. This allows us to describe geometry through mathematics.

The algebraic formulation can be derived from the differential formulation through an integration process (Fig. 5) that is needed because, while in differential formulation geometry must be eliminated from the physical laws, in the numerical solution geometry is essential.

The Finite Volume Method and the Finite Difference Method are also based on a differential formulation (Fig. 5). The Cell Method, on the contrary, uses global variables and balance equations in a global form. As a consequence, the relative governing equations are expressed directly in algebraic form.

The CM uses cell-complexes (Fig. 5), which are the generalization of the coordinate systems to the algebraic formulation. The CM cell-complexes are not actually the result of a domain discretization, a process needed in numerical analysis, as in the case of the FEM. They are required in algebraic formulation, since global variables are associated not only with points, as for differential formulation, but also with lines, surfaces and volumes. In the CM, global variables are described directly as they will be associated with the related space elements of the cell-complexes. Consequently, in the algebraic formulation of physics, cell complexes have the same role that coordinate systems have in differential formulation. Physical notions are therefore translated into mathematical notions through the intermediation of topology and geometry.

The geometrical structure of space is very rich in algebraic formulation. It is possible, for example, to define an inner orientation for the elements in dimension 0, 1, 2 and 3 of a cell-complex (Fig. 5), which we call the primal cell-complex. Then, by considering the planes that are equidistant from the primal nodes, we can define a second cell complex (Fig. 5), called the dual cell-complex, which turns out to be provided with an outer orientation.

Moreover, if we consider a time axis and subdivide a given time interval into many adjacent small time intervals (Fig. 5), we have a primal cell complex in time. In order to build the dual cell complex in time, we will consider the middle instant of each time interval. The result is that, similarly for the space elements, the primal and dual time elements are also provided with inner and outer orientation.

We will now see how the space elements of the primal and dual cell-complexes are strictly associated with global variables. For the example, in Fig. 6, on a two-dimensional domain, once a mesh has been introduced, it is natural to associate the primal nodes with the displacements of the primal nodes and the total load over an area surrounding the primal nodes, which is an area of the dual cell complex. It follows that the displacements, which are configuration variables, are computed on objects of the primal mesh, while the loads, which are source variables, are computed on objects of the dual complex.

This result is general, independently of the kind of configuration or source variable, the shape of the domain, or the physical theory involved. In effect, for each set of primal nodes and for each given physical theory, the source variables are always associated with the elements of the dual cell-complex and the configuration variables are always associated with the elements of the primal cell-complex (Fig. 7).

The association of physical variables with the elements of a cell complex and its dual was first introduced by Okada and Onodera (1951) and Branin (1966). In the CM, the strong coupling between physical variables and oriented space elements

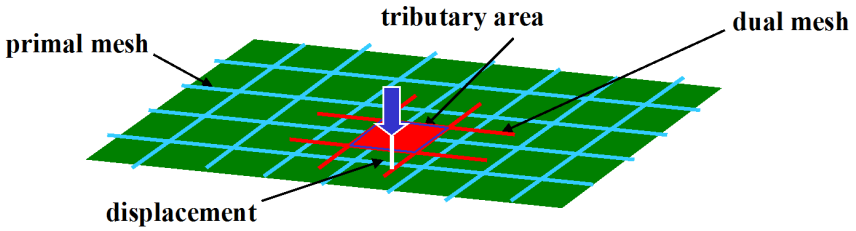

Figure 6: Association between global variables and elements of the two cell complexes

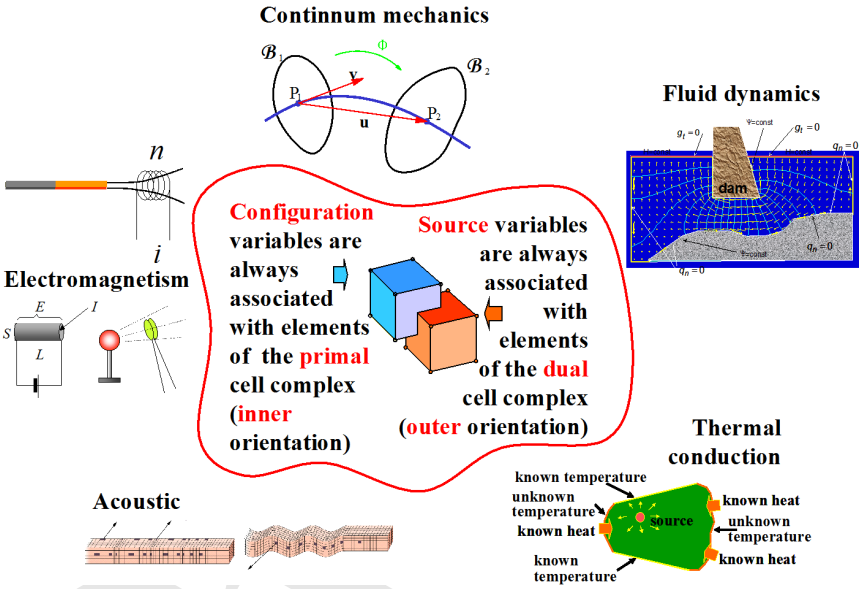

Figure 7: Association between global variables and space elements of the primal and dual cell complexes, in different physical theories

becomes the key to giving a direct discrete formulation to physical laws.

Moreover, the existence of an underlying structure, common to different physical theories, is mainly responsible for the structural similarities presented in physical theories, commonly called “analogies”. Today, we are able to explain these analogies in the light of the association between the global variables and the four space elements, since the homologous global variables of two physical theories are those associated to the same space element. In other words, the analogies between physical theories arise from the geometrical structure of the global variables and not

from the similarity of the equations that relate variables to each other in different physical theories [Tonti (in press)].

3 The structure of the governing equations in the CM

3.1 Nonlocality in algebraic and differential formulations

For the nodes of a dual cell-complex in plane domains, we can choose, for example, the barycenters of a primal cell-complex made of triangles, a simplicial primal mesh (Fig. 8).

Since even in plane domains the primal mesh has thickness, which is a unit thickness, the dual nodes are not in the same plane as the primal nodes and the two meshes are shifted along the thickness (Fig. 9).

This is similar to the relative position of staggered elements, a commonly used mathematical expedient to avoid spurious solutions in physics. In particular, in solid mechanics, imbricate elements [Bažant, Belytschko, and Chang (1984); Belytschko, Bažant, Hyun, and Chang (1986)] are examples of staggered elements used for regularizing material instability in strain-softening materials, both in one- and two-dimensional domains (Fig. 10).

As it is now clear that global physical variables are naturally associated with the space elements of the primal and dual cell-complexes, it therefore follows that staggering is not only a mathematical expedient used to regularize the solution [Stevens and Power (2010)], but is also necessary in physics to take account of the association between physical variables and oriented space elements.

In solid mechanics, staggering is used to provide the differential formulation with nonlocal properties, when modeling heterogeneous materials. It is now the common opinion that the classical local continuum concept, where stress at a given point depends only on the deformation and temperature history at that precise point, cannot adequately describe damage in heterogeneous materials by means of differential formulation, particularly when size-effect is involved. Indeed, modeling the size-effect is impossible in the context of classical plasticity, both in problems involving strain-softening [Duhem (1893); Krumhansl (1965); Rogula (1965); Eringen (1966); Kunin (1966); Kröner (1968)] and in those with no strain-softening at all.

The first criticisms of the local approach date back to the 1960s [Krumhansl (1965), Rogula (1965), Eringen (1966), Kunin (1966) and Kröner (1968)] and are based on the microstructure of matter. In effect, all materials are characterized by microstructural details, with size ranging over several orders of magnitude [Bažant and Jirásek (2002)]. They cannot, therefore, be broken down into a set of infinites-

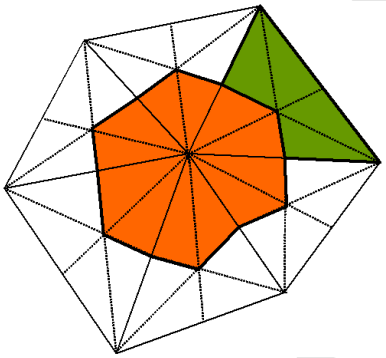

Figure 8: Primal and dual cell complexes in plane domains

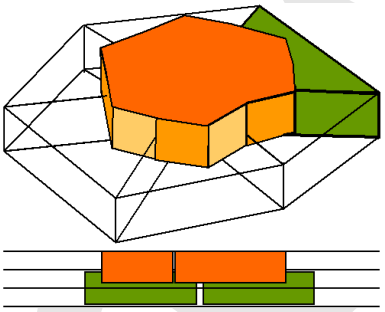

Figure 9: Staggering of the primal and dual cell complexes along the thickness

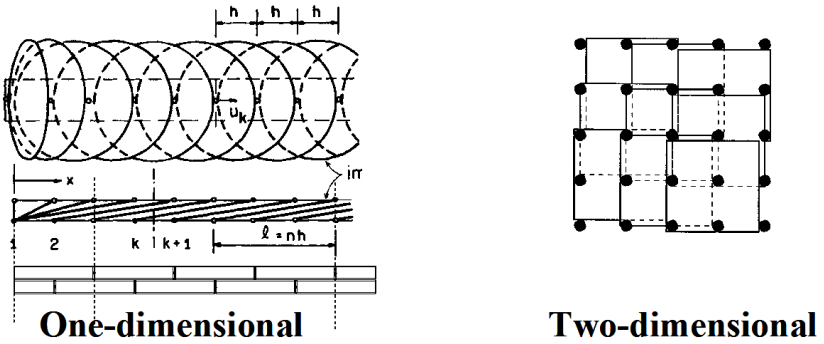

Figure 10: Imbricate elements in one- and two-dimnsional domains

imal volumes, each of which can be described independently. Consequently, the idea was advanced that heterogeneous materials should be modeled properly by some kind of nonlocal continuum [Duhem (1893)], in which the stress at a certain point is a function of the strain distribution over a certain representative volume centered at that point [Bažant and Chang (1984)]. This idea led to models where the classical continuum description is improved by introducing an internal length parameter into the constitutive laws.

According to the mathematical definition of nonlocality in the narrow sense, given by Rogula, the operator A in the abstract form of the fundamental equations of any physical theory:

$$Au = f, \tag{1}$$

is called local when, if $u(x) = v(x)$ for all x in a neighborhood of point x_0, then $Au(x) = Av(x)$. Bažant and Jirásek (2002) and Ferretti (2005) pointed out that the differential operators satisfy this condition, because the derivatives of any arbitrary order do not change if the differentiated function only changes outside the small neighborhood of the point where the derivatives are taken, and, consequently, the differential operators are local. It follows that any formulation using differential operators is intrinsically local. That is, differential formulation is not adequate for describing nonlocal effects.

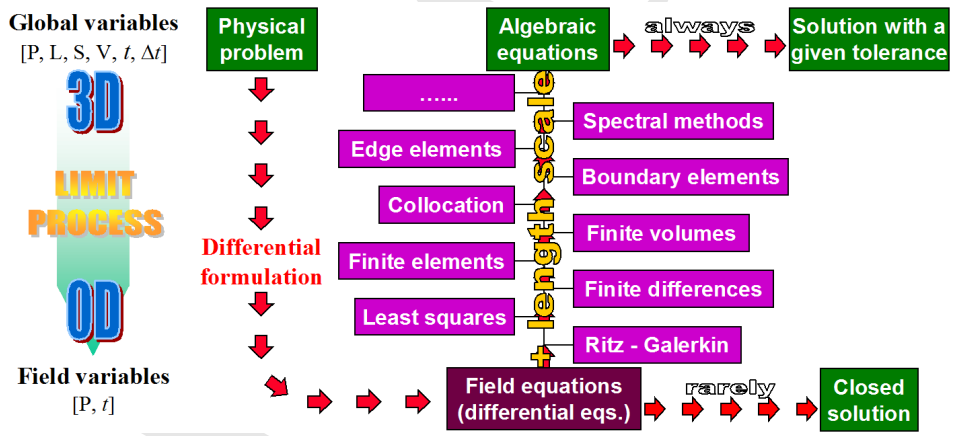

Figure 11: Losing and reintroducing metrics in the differential formulation

In the light of the former discussion on the geometrical content of global variables, we can now provide an alternative interpretation of nonlocality. As we have discussed previously, the reason why differential operators are local in nature lies in

the use of the limit process. The density-finding process is carried out with the intention of formulating the field laws in an exact form. However, differential formulation can be solved only for very simple geometries and under particular boundary conditions (Fig. 11). Moreover, with the global variables being reduced to point and instant variables, we can no longer describe more than 0-dimensional effects, that is, the nonlocal effects. Metrics must be reintroduced a-posteriori in the discretization process, if we want to model nonlocality (Fig. 11).

We may now ask where the length scale is to be reintroduced. In nonlocal approaches, a length scale is incorporated into the constitutive laws, but there is no evidence that this choice is the only one, or even the most appealing from a physical point of view. On the contrary, we have seen that the physical global variables themselves have a multi-dimensional geometrical content. It therefore seems that dimensional scales and nonlocal effects are directly associated with global variables, and nonlocality seems to be a property of global variables, not a prerogative of constitutive laws. Consequently, reintroducing or preserving nonlocality in governing equations is physically more correct than reintroducing nonlocality into constitutive equations. When speaking of reintroduction, we are dealing with differential formulation, while, when speaking of preservation, we are dealing with algebraic formulation. The difference is not negligible, since, in order to reintroduce a length scale, a suitable approach must be developed, while, in order to preserve the length scales, it is sufficient to avoid the limit process and, by using an algebraic approach, a nonlocal formulation is automatically obtained.

Besides this, due to the structure of the discrete operators, even the topological equations are provided with nonlocal properties in algebraic formulation. The reason for this lies in the relationship between the discrete p-forms of different degrees.

3.2 The discrete p-forms (cochains)

A physical variable ϕ associated with one set of p-cells of a cell-complex defines a discrete p-form (or a discrete form of degree p). The potential of a vector field, line integral of a vector, flux and mass content are discrete forms of degree 0, 1, 2 and 3, respectively (Tab. 1).

Table 1: Examples of discrete p-forms

Variable	Potential of a vector field	Line integral of a vector	Flux	Mass content
Evaluated on	0-cells (points)	1-cells (lines)	2-cells (surfaces)	3-cells (volumes)
Discrete *p*-form	discrete 0-form	discrete 1-form	discrete 2-form	discrete 3-form

The discrete p-forms generalize the notion of field functions, because, in a discrete p-form $\Phi[S]$, we associate the value of a physical variable to the space elements of degree p, while the field functions $f(P)$ always associate the value of a physical variable to the points of the domain. As a consequence, $\Phi[S]$ is a set function, while $f(P)$ is a point function.

The notion of discrete form is the discrete version of the exterior differential form, a mathematical formalism that has the great merit of highlighting the geometrical background of physical variables, something ignored by differential calculus, providing a description that is independent of the coordinate system used. Nevertheless, this formalism uses field variables instead of global variables and, for this reason, it must use the notion of derivative.

3.3 The coboundary process and its implications on nonlocality

The coboundary process on a discrete p-form is a process that generates a discrete $(p+1)$-form. It is worth noting that balance, circuital equations, and equations forming differences can be expressed by the coboundary process performed on discrete p-forms of degree 2, 1, 0, respectively. Thus, the coboundary process plays a key role in physics.

This process is analogous, in an algebraic setting, to the exterior differentiation on exterior differential forms and leads to discrete operators, whose elements are the incidence numbers, equal to 0, +1, or -1. In particular, the incidence number of a p-cell with a $(p-1)$-cell is equal to:

- 0, if the $(p-1)$-cell is not on the boundary of the p-cell;
- +1, if the $(p-1)$-cell is on the boundary of the p-cell and the orientations of the p-cell and $(p-1)$-cell are compatible (Fig. 12);
- -1, if the $(p-1)$-cell is on the boundary of the p-cell and the orientations of the p-cell and $(p-1)$-cell are not compatible (Fig. 13).

In a three-dimensional space, we can define three incidence matrices:

- **G**: matrix of the incidence numbers between 1-cells and 0-cells;
- **C**: matrix of the incidence numbers between 2-cells and 1-cells;
- **D**: matrix of the incidence numbers between 3-cells and 2-cells.

The incidence matrices **G**, **C**, and **D** are the discrete versions of the differential operators "grad", "curl", and "div", respectively.

The coboundary process on a discrete p-form, Φ^p, is performed in two steps:

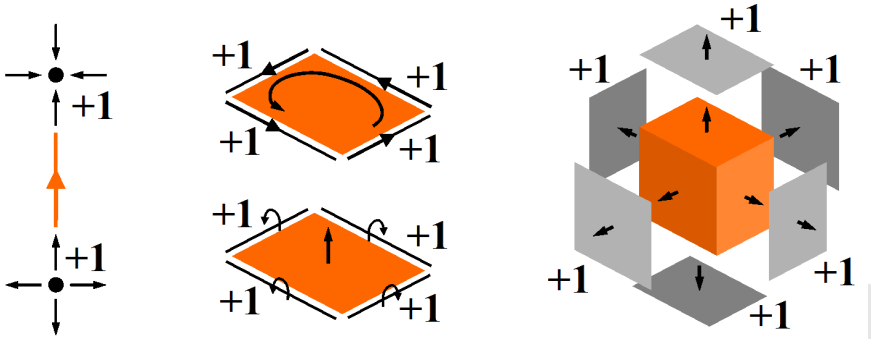

Figure 12: Incidence numbers equal to +1

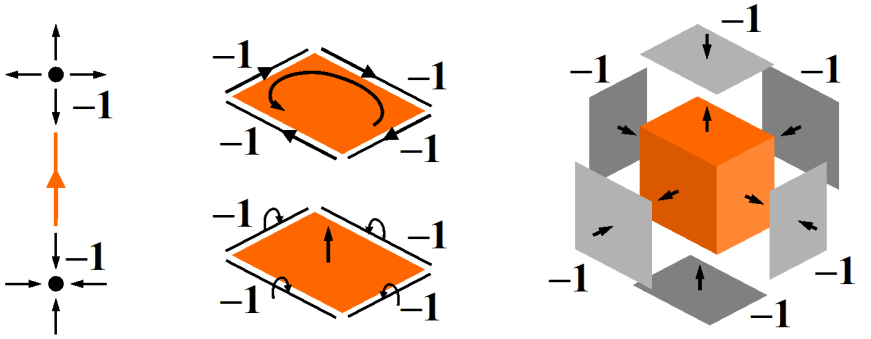

Figure 13: Incidence numbers equal to -1

1. For each p-cell: we assign the value ϕ_n^p, evaluated on the *nth* p-cell, to each coface (of degree p+1) of the *nth* p-cell, with the plus or minus sign according to the mutual incidence number;

2. For each (p+1)-cell: we sum the values $\phi^p_{1,2,\ldots,h}$ coming from all the faces (of degree$p$) of the boundary.

At the end of the process, we obtain a discrete (p+1)-form.

The consequence of this two-step process, when we enforce balance in the CM, which is a coboundary process on a discrete p-form of degree 2, the flux, described as follows:

- first step (on each 2-cell): transfer the fluxes associated with all 2-cells to their cofaces, each multiplied by the relative incidence number;

- second step (on each 3-cell): perform the algebraic sum of the fluxes coming from the first step;

is that the global source variables involved in the balance of one 3-cell are also involved in the balance of all the surrounding 3-cells. This happens since each 3-cell is common to more than one 2-cell. Consequently, the balance at a given 3-cell does not depend on the current values, or previous history, of the global source variables at that 3-cell only, but on the current values, and previous history, taken by the global source variables in all the surrounding 3-cells. This gives nonlocal properties to the balance equations of the algebraic formulation, while the balance equations of the differential formulation are local.

Achieving nonlocality in the CM balance equations is very important, as it enriches the description of physics given by the CM, compared to the descriptions given by any other numerical method, even those known as discrete methods. This enrichment follows on from the structure of the coboundary process given to the balance equations and from more than one 2-cell sharing the same 3-cell. That is, it is a consequence of the structure of the balance equations, and there is no need to modify the balance equations in any way to provide them with nonlocal properties.

Moreover, coboundary processes on discrete p-forms defined on configuration variables generate discrete $(p+1)$-forms defined on configuration variables, and coboundary processes on discrete p-forms defined on source variables generate discrete $(p+1)$-forms defined on source variables. Thus, the coboundary process creates a relationship between different degrees of discrete forms of the same kind of variable. This means that all the topological equations are coboundary processes [Ferretti (in press)], and therefore, even enforcing compatibility in the CM is a coboundary process. As a consequence, even the Kinematic equations are provided with nonlocal properties in the CM, while the Kinematic equations of the differential formulation are local. We can therefore conclude that all the governing equations of the algebraic formulation are nonlocal: constitutive relationships are nonlocal due to the staggering between the elements of the primal cell complex, on which we compute configuration variables, and the dual cell complex, on which we compute source variables, while the balance and Kinematic equations are nonlocal, since they are the results of coboundary processes. This means that obtaining a nonlocal formulation by using discrete operators is possible, besides being physically appealing. The new nonlocal formulation is desirable from a numerical point of view, since the numerical solution is reached sooner using discrete operators than with differential operators.

3.4 Implementation of the equations

The CM equations are implemented in the same manner as for FEM. The linear interpolation of the CM for solid mechanics in two-dimensional domains was provided in Ferretti (2003a), with a Kinematic equation for each primal cell:

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{u}, \tag{2}$$

$$\boldsymbol{u} = \begin{bmatrix} w_k & v_k & w_i & v_i & w_j & v_j \end{bmatrix}^T, \tag{3}$$

$$\boldsymbol{B} = \frac{1}{\Delta_{ijk}} \begin{bmatrix} \Delta y_{ji} & 0 & \Delta y_{kj} & 0 & \Delta y_{ik} & 0 \\ 0 & \Delta x_{ij} & 0 & \Delta x_{jk} & 0 & \Delta x_{ki} \\ \Delta x_{ij} & \Delta y_{ji} & \Delta x_{jk} & \Delta y_{kj} & \Delta x_{ki} & \Delta y_{ik} \end{bmatrix}. \tag{4}$$

a constitutive law:

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}. \tag{5}$$

and an equilibrium equation for each dual cell (Fig. 14):

$$\sum\nolimits_j \boldsymbol{Q}_{kj} + \boldsymbol{F}_k = 0, \tag{6}$$

$$\boldsymbol{Q} = \boldsymbol{N}\boldsymbol{\sigma} = \boldsymbol{N}\boldsymbol{D}\boldsymbol{\varepsilon} = (\boldsymbol{N}\boldsymbol{D}\boldsymbol{B})\boldsymbol{u}, \tag{7}$$

$$\boldsymbol{N} = \begin{bmatrix} S_x & 0 & S_y \\ 0 & S_y & S_x \end{bmatrix}, \tag{8}$$

$$\boldsymbol{S} = \boldsymbol{R}\boldsymbol{L}. \tag{9}$$

The linear system of equations can be written in the form:

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}, \tag{10}$$

where $\boldsymbol{F}$ and $\boldsymbol{U}$ are the force and displacement vectors, respectively, and $\boldsymbol{K}$ is analogous to the FEM stiffness matrix, which is symmetric and defined as positive for properly constrained systems.

The CM was also implemented for solid mechanics with quadratic interpolation of the displacements in two-dimensional domains [Cosmi (2000)] and three-dimensional domains [Pani and Taddei (2013)].

Lastly, Zovatto (2001) proposed a meshfree approach of the CM, including for three-dimensional domains. Meshfree and meshless approaches are very useful in problems of fracture mechanics where the crack being studied is simulated as a discontinuity of the displacement field. In effect, crack geometry updating and remeshing on the whole domain is a very expensive process from a computational

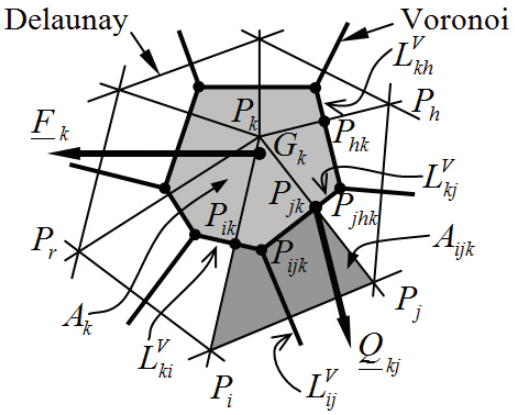

Figure 14: Primal and dual meshes for CM analysis in two-dimensional domains

point of view. Some of the most recent achievements for the CM meshless approach can be found in Pani and Taddei (2013) and Taddei, Pani, Zovatto, Tonti, and Viceconti (2008).

As far as the convergence rate is concerned, it has been shown [Tonti (2001)] that this depends on the choice of dual polygons. The most convenient choice, giving a convergence rate equal to four [Cosmi (2000)], is to use Gauss points to build the dual polygons.

4 Conclusions

In the present paper, we have discussed where the Cell Method stands in respect to the discussion on local or nonlocal descriptions of the continuum for modeling heterogeneous brittle materials. We have found that the CM can provide a direct nonlocal description of the continuum, without requiring any sort of enrichment to the constitutive laws, by means of length scales, as is usually the case for nonlocal approaches in solid mechanics. In particular, we can state that the CM does not provide an enriched continuum description only as far as the constitutive relationships are concerned. It enriches all the governing equations involved in the physical theory in a very simple manner, by simply taking account of the association between global variables and extended space elements. The consequence is that, by using the CM, there is no need to recover nonlocality a-posteriori, as for differential formulation. Nonlocality is – we could say – intrinsic to algebraic formulation and is the result of using global instead of field variables, something that distinguishes the cell method from any other numerical method, at the moment.

As already discussed in Ferretti [2005], it is worth noting that, in the first theories of nonlocal elasticity developed by Eringen and Edelen [Eringen (1966); Edelen et al. (1971); Eringen (1972); Eringen and Edelen (1972)], nonlocality was a property of the elastic problem in its complex, and not solely of its constitutive relationships. In other words, in these nonlocality theories, there was already the idea that nonlocality is a property of the governing equations. Nevertheless, this idea was not developed further, since the theories of nonlocal elasticity were too complicated to be calibrated and verified experimentally, let alone to be applied to any real problems [Bažant and Jirásek (2002)]. Treating only the stress-strain relationships as nonlocal, while the equilibrium and kinematic equations and their corresponding boundary conditions retain their standard form, was something needed later [Eringen and Kim (1974); Eringen et al. (1977)], to provide a practical formulation of these early theories. Consequently, incorporating the length scale into the constitutive relationships only is the practical simplification of a more general theory and has no evident justification from a physical point of view. In this sense, we can state that the cell method provides a physically more appealing nonlocal formulation when compared to nonlocal differential approaches. Ferretti provides several numerical results [Ferretti (2012), Ferretti (2005), Ferretti (2004c) and Ferretti (2003b)] showing how the CM, together with a new local constitutive law for heterogeneous brittle materials – the effective law [Ferretti (2004d); Ferretti (2004e); Ferretti (2004f); Ferretti and Di Leo (2008)] – actually offers a nonlocal description for solid mechanics, allowing us to model the size- and shape-effects, which is impossible in local differential approaches.

A properly formulated enriched classical continuum for the differential formulation has a regularizing effect when modeling strain-softening materials, because it acts as a localization limiter, so that the boundary value problem is once again well-posed. Boundary value problems with strain-softening constitutive models that are ill-posed are just some of the many examples of spurious solutions for differential formulation. Since the CM is a multidisciplinary method, the intrinsic nonlocality of the CM governing equations means that we can assume that, using the CM, spurious solutions can be avoided whenever they appear in differential formulation, independently of the physical theory involved. This is the really strong point for the CM, setting it apart from all other numerical methods, for the moment.

In conclusion, the Cell Method enriches the computational description of physics by using notions of algebraic topology, as well as of mathematics. The result is twofold:

- The Cell Method provides the physical laws in an algebraic manner, directly, avoiding the use of any discretization procedure.

- The Cell Method is tantamount to abandoning the principle of local action, therefore of avoiding the spurious solutions of differential formulation.

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