Modelling and analysis of Timoshenko beams weakened by multiple transverse open cracks

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SOMMARIO Il lavoro presenta un metodo per l'analisi delle vibrazioni libere di travi di Timoshenko soggette a sollecitazione flesso-torsionale in presenza di più fessure. Il metodo si basa sull'uso congiunto della matrice di rigidezza dinamica e della "line-spring" per la modellazione della sezione fessurata. Viene proposto uno studio parametrico sulla risposta modale al variare della posizione e della profondità delle fessure.

ABSTRACT In this paper an exact solution methodology, based on the coupling of the dynamic stiffness matrix and the line-spring, enabling one to analyze the coupled bending-torsion free vibration of Timoshenko beams weakened by multiple non-propagating part-through surface cracks is presented. The changes introduced by the presence of three transverse open cracks, regarding the modal response, are investigated. A parametric study has been carried out for various crack parameters such as crack depth and location.

1. PRINCIPLES OF THE METHOD

A straight uniform beam element of length L and T-cross-section is shown in Fig. 1, with the mass axis and the elastic axis (i.e. the loci of the mass centre and the shear centre of the cross-section) being separated by a distance x_α. In the right-handed coordinate system of Fig. 1, the elastic axis, which is assumed to coincide with the *y*-axis, is permitted flexural translation $\bar{h}(y,t)$ in the *z*-direction and torsional rotation $\psi(y,t)$ about the *y*-axis, where *y* and *t* denote distance from the origin and time, respectively. The governing partial differential equations of motion for the coupled bending-torsional free natural vibration of the Timoshenko beam are given by

$$EI\theta'' + kAG\left(\bar{h}' - \theta\right) - \rho I\ddot{\theta} = 0 \tag{1}$$

$$kAG\left(\bar{h}'' - \theta'\right) - m\left(\ddot{\bar{h}} - x_\alpha\ddot{\psi}\right) = 0 \tag{2}$$

$$GJ\psi'' - I_\alpha\ddot{\psi} + mx_\alpha\ddot{\bar{h}} = 0 \tag{3}$$

where: *E* is the Young's modulus, *G* is the shear modulus and *ρ* is the density of the material; *EI*, *GJ* and *kAG* are, respectively, the bending, torsional and shear rigidities of the beam; *I* is the second moment of area of the beam cross-section about the *x*-axis, *k* is the section shape factor, *A* is the cross-section area, $m = \rho A$ is the mass per unit length, I_α is the polar mass moment of inertia per unit length about the *y*-axis (i.e. an axis through the shear centre), *θ* is the angle of rotation in radians of the cross-section due to the bending alone (so that the total slope $\bar{h}'$ equals the sum of slope due to bending and due to shear deformation) and primes and dots denote differentiation

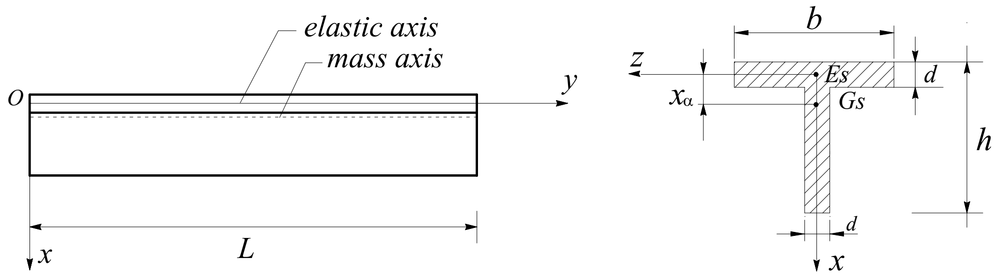


Figure 1. Coordinate system for a coupled bending-torsional Timoshenko uncracked beam with T cross-section.

with respect to position y and time t, respectively. Equations (1)-(3) together with appropriate end conditions completely define the coupled bending-torsional free vibration of a Timoshenko beam.

If a sinusoidal variation of $\bar{h}$, θ and ψ, with circular frequency ω, is assumed, then the amplitude of $\bar{h}$, θ and ψ are obtained in terms of a set of arbitrary constants. Next, the arbitrary constants are eliminated by imposing the prescribed end conditions for the beam displacements H_1, Θ_1, Ψ_1 and H_2, Θ_2, Ψ_2 and forces S_1, M_1, T_1 and S_2, M_2, T_2. By extensive algebraic manipulation, the solution for the bending displacement $H(\xi)$, bending rotation $\Theta(\xi)$ and torsional rotation $\Psi(\xi)$ are obtained from the following equations:

$$H(\xi) = A_1 cosh\alpha\xi + A_2 senh\alpha\xi + A_3 cos\beta\xi + A_4 sen\beta\xi + A_5 cos\gamma\xi + A_6 sen\gamma\xi \tag{4}$$

$$\Theta(\xi) = B_1 sinh\alpha\xi + B_2 cosh\alpha\xi + B_3 sin\beta\xi + B_4 cos\beta\xi + B_5 sin\gamma\xi + B_6 cos\gamma\xi \tag{5}$$

$$\Psi(\xi) = C_1 cosh\alpha\xi + C_2 senh\alpha\xi + C_3 cos\beta\xi + C_4 sen\beta\xi + C_5 cos\gamma\xi + C_6 sen\gamma\xi \tag{6}$$

where $A_1 - A_6$, $B_1 - B_6$ and $C_1 - C_6$ are the three different sets of constants, $\xi = y/L$ and α, β, γ are constants reported in [1]. The expressions for the bending moment $M(\xi)$, the transverse force $S(\xi)$ and the torque $T(\xi)$ can be obtained from eqs. (4)-(6).

The dynamic stiffness matrix which relates the amplitudes of the sinusoidally varying forces **F** to the corresponding displacement amplitudes **U** can be derived and represented in a compact form as

$$\mathbf{F} = \mathbf{KU} \tag{7}$$

where **K** is the required stiffness matrix.

2. CRACK MODELLING

In order to study the behaviour of a cracked structure a suitable model of the cracked section is required. In the present study the transverse cracks have been considered as open. Hereinafter, the cracked sections are represented as an elastic hinge [2] with spring constants simulating flexional, shear and torsional deformations. Bending moment M, shearing force S and torsion T can be related to the bending rotation Θ, the deflection H and the torsional rotation Ψ by the following relations: $\Theta = \lambda_{mm} M$, $H = \lambda_{ss} S + \lambda_{st} T$, $\Psi = \lambda_{tt} T + \lambda_{ts} S$, where λ_{mm}, λ_{ss}, λ_{tt} and $\lambda_{st} = \lambda_{ts}$ are compliance expressions for bending, shear and torsion, respectively, as a function of the stress intensity factors K_I, K_{II}, K_{III} for Mode I, II and III, respectively. By means of Castigliano's theorem, the stiffness matrix $\mathbf{k}_f$ for the cracked element may be derived. If multiple cracks are present in the T-beam, as considered in this paper, the structure is divided into substructures, the number of which depends on the number of cracks considered in the structure itself, characterized by dynamic stiffness matrices, on the left and on the right of the cracked section, plus one line-spring element to model each cracked section. It should be noted that for the T cross-section considered in this paper, the estimate of the stress intensity factors is based on the simple method proposed in [3].

The global dynamic stiffness matrix for the whole structure can be assembled using the above dynamic stiffness matrices of all substructures and the line spring stiffness matrix, by applying the standard procedure of the finite element method.

3. EIGENVALUES AND MODE SHAPES OF THE CRACKED BEAM

Once the global dynamic stiffness matrix of the system is obtained, after introducing the boundary conditions at the ends of the beam, one finally obtains the frequency equation. The restrained global stiffness matrix is denoted by $\mathbf{K_g}^*(\omega)$. The natural frequencies are those values of ω for which

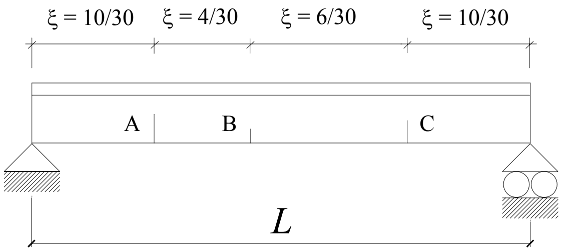
$$\mathbf{K_g}^*(\omega)\mathbf{A_g}^*=\mathbf{0} \tag{9}$$

where $\mathbf{A_g}^*$ is the restrained vector of constants which allow us to define the modal shapes, namely the vector of the nodal displacement amplitudes. The necessary and sufficient condition for nonzero elements in the column vector $\mathbf{A_g}^*$ of Eq. (9) is that $\Delta = \left|\mathbf{K}_\mathbf{g}^*(\omega)\right|$ shall be zero, and the vanishing of Δ determines the natural frequencies of the system. The roots of the non-linear equation were obtained by using the Matlab function "fzero". Once the cracked frequencies are found, one can obtain the cracked mode shapes of the beam using Eqs. (5)-(7). The above functions must also satisfy certain conditions at the cracked sections or at the ends of each substructures.

4. NUMERICAL EXAMPLE

The changes introduced by the presence of three cracks, regarding the magnitude of natural frequencies of a structure as well as its modal response, are investigated. A parametric study of three transverse open cracks has been carried out for various crack parameters such as crack depth and crack location. As shown in Fig. 3, the simple beam contains three initial notches. By varying the sizes of these initial notches, various cracking behaviours are obtained. In the example, notches A and B are assigned the size of a/h = 0.4 and a/h = 0.2, respectively. Notch C is enlarged from a/h = 0.2 to a/h = 0.4. Fig. 3 illustrates the first four bending modal shapes, as a function of the crack sizes. When notch C is enlarged from a/h = 0.2 to a/h = 0.4, the first three modal shapes tend to move toward right with the increasing of the size crack C and the first and the fourth modal shape are different between them depending on the size of the initial notches A and B.

There is no changes in the second and third modal shapes because the notch B is very close to the nodal point and the only important change is in the fourth mode. Thus, small changes in the sizes or positions of initial notches may completely alter the modal shapes, which makes it very difficult to predict the cracking behaviour in a real structure when multiple cracks are involved.



L = 1m	x_α = 0.0111 m
h = 0.05 m	I_α = 3.237 x 10^{-3} kgm
b = 0.05 m	EI = 41241.669 Nm^2
m = 7.02 kg/m	GJ = 2368.421 Nm^2
d = 0.01 m	

Figure 2. Geometry, Cartesian coordinate systems and all geometric details of the beam with three edge cracks and T-section.

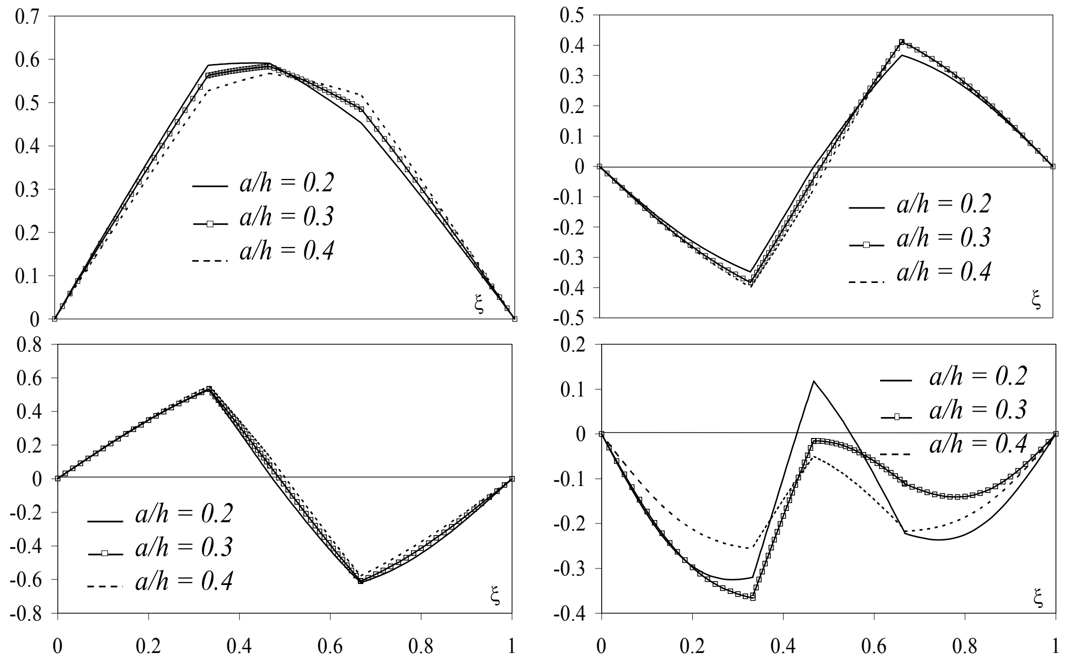


Figure 3. First four bending modal shapes.

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