Similarities between Cell Method and Non-Standard Calculus

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Abstract: - Infinitesimal analysis has without doubt played a major role in the mathematical treatment of physics in the past, and will continue to do so in the future, but we must also be aware that several important aspects of the phenomenon being described, such as its geometrical and topological features, remain hidden, in using the differential formulation. This is a consequence not of performing the limit, in itself, but rather of the numerical technique used for finding the limit. In this paper, we analyze and compare the two most known techniques, the iterative technique and the application of the Cancelation Rule for limits. It is shown how the first technique, leading to the approximate solution of the algebraic formulation, preserves information on the trend of the function in the neighbourhood of the estimation point, while the second technique, leading to the exact solution of the differential formulation, does not. Under the topological point of view, this means that the algebraic formulation preserves information on the length scales associated with the solution, while the differential formulation does not. This new interpretation of the Cancelation Rule for limits is also discussed in the light of the findings of non-standard calculus, the modern application of infinitesimals, in the sense of non-standard analysis, to differential and integral calculus.

Key-Words: - Differential Formulation, Algebraic Formulation, Cell Method, Non-Standard Calculus, Enriched Continuum, Numerical Stability.

1 Introduction

In this paper, we analyze the difference between the algebraic and the differential formulation from the mathematical point of view. Particular attention is devoted to the computation of limits, by highlighting how the numerical techniques used for performing limits may imply a loss of information.

The main motivation for the most commonly used numerical technique in differential formulation, the Cancelation Rule for limits, is to avoid the iterative computation of limits, which is implicit in the definition itself of a limit (the $\varepsilon-\delta$ definition of a limit). The reason for this is that iterations necessarily involve some degree of approximation, while the purpose of the Cancelation Rule for limits is to provide a direct exact solution. Nevertheless, this exact solution is only illusory, since we pay the direct computation of the Cancelation Rule for limits by losing information on the trend of the function in the neighbourhood of the estimation point. Conversely, by computing the limit iteratively, with the dimension of the neighbourhood that decreases at each iteration, leading also the error on the solution to decrease, we conserve information on the trend of the function in the neighbourhood of the estimation point. This second way to operate, where the dimension of the neighbourhood approaches zero but is never equal to zero, follows from the $\varepsilon-\delta$ definition of a limit directly and leads to the algebraic formulation.

When the Cancelation Rule for limits is used for finding densities and rates, we also lose information on the space and time extent of the geometrical and temporal objects associated with the variables we are computing, obtaining point- and instant-wise variables. By using the algebraic formulation, on the contrary, we preserve both the length and the time scales. Consequently, the physical variables of the algebraic formulation maintain an association with the space and time multi-dimensional elements.

The Cancelation Rule for limits acts on the actual solution of a physical problem as a projection operator. The consequence is that the algebraic formulations is to the differential formulation as the actual solution of a physical problem is to the projection of the actual solution on the tangent space of degree 0, where each physical phenomenon is described in terms of space elements of degree 0, the points, and time elements of degree 0, the time instants. In other words, the differential solution is the shadow of the algebraic solution in the tangent space of degree 0.

We also discuss how using the algebraic formulation, instead of the differential formulation, is similar to performing non-standard calculus, the modern application of infinitesimals to differential and integral calculus, instead of standard calculus. In this sense, the derivative of a function can be viewed as the standard part, or the shadow, of the difference quotient. The extension of real numbers, which leads to non-standard calculus, is indeed an attempt to recover the loss of length scales. Specifically, the enrichment with a length scale has a regularization effect on the solution.

2 A Discussion on How to Compute Derivatives

The $\varepsilon-\delta$ definition of a limit is the formal mathematical definition of a limit. Let f be a real-valued function defined everywhere on an open interval containing the real number c (except possibly at c) and let L be a real number. According to the $\varepsilon-\delta$ definition of a limit, the statement

$$\lim_{x\to c} f(x) = L\,, \tag{1}$$

means that, for every real $\varepsilon>0$, there exists a real $\delta>0$ such that, for all real x, if $0<|x-c|<\delta$, then $|f(x)-L|<\varepsilon$. Symbolically:

$$\forall\varepsilon>0$$
$$\exists\delta>0:\ \forall x\big(0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon\big) \tag{2}$$

The absolute value $|x-c|$ in Eq. (2) means that x is taken sufficiently close to c from either side (but different from c). The limit value of $f(x)$ as x approaches c from the left, $x\to c^-$, is denoted as left-hand limit, and the limit value of $f(x)$ as x approaches c from the right, $x\to c^+$, is denoted as right-hand limit. Left-handed and right-handed limits are called one-sided limits. A limit exists only if the limit from the left and the limit from the right are equal. Consequently, the limit notion requires a smooth function.

The derivative $f'(x)$ of a continuous function $f(x)$ is defined as either of the two limits (if they exist):

$$f'(x) \triangleq \lim_{s\to x}\frac{f(s)-f(x)}{s-x}\,, \tag{3}$$

and

$$f'(x) \triangleq \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,,\ h>0\,. \tag{4}$$

The ratio in Eq. (4) is not a continuous function at $h=0$, because it is not defined there. In fact, the limit (4) has the indeterminate form $(\to 0)/(\to 0)$ as $h\to 0$, since both the numerator and the denominator approach 0 as $h\to 0$.

We can compute the limiting value (4) both in an approximated way, by reducing the error with subsequent iterations, as per the $\varepsilon-\delta$ definition of a limit, or in an exact way, by making use of the Cancellation Rule for limits. In the perspective of a computational analysis using the differential formulation, it is obvious that the choice falls on the exact, rather than the approximated, computation of limits. In effect, in doing so, one can obtain an exact solution of the physical phenomenon under consideration only in few elementary cases, with simple geometric shapes of the domain and under particular boundary conditions. Anyway, the most important aspect is not that the exact numerical solution is hardly ever attained in real cases, but rather, that the choice itself of the term “exact” for the limit promised by the Cancelation Rule is not entirely appropriate. Actually, in order to provide the solution of the limit directly, the Cancellation Rule for limits reduces the order of zero both in the numerator and the denominator by one. Under the numerical point of view, this reduction is made by cancelling a quantity with the order of a length, both in the numerator and in the denominator. Under the topological point of view, we could say that the reduction degrades the solution, in the sense that, being deprived of one length scale, the solution given by the Cancelation Rule provides us with a lower degree of detail in describing the physical phenomenon under consideration. Specifically, with the Cancelation Rule for limits, we can factor h out of the numerator in Eq. (4):

$$\big[f(x+h)-f(x)\big]\Big|_{x=\bar{x}} = h\cdot g(h)\,, \tag{5}$$

and cancel this common factor in the numerator and denominator. Then, we can find the limit by evaluating the new expression at $h=0$, that is, by plugging in 0 for h, because the new expression is continuous at $h=0$:

$$\lim_{h\to 0}\frac{h\cdot g(h)}{h} = g(h)\Big|_{h=0}\,, \tag{6}$$

where the result is a real number.

The equality in Eq. (6), established by the Cancelation Rule, is undoubtedly numerically correct, in the sense that the results of the left- and right-hand-side expressions are actually numerically equal, but the way in which these results are achieved is radically different in the two cases. As a matter of fact, the limit on the left side is defined on the open interval of length h, while the function $g(h)$ is evaluated for a given value of the variable, $h = 0$. This difference, negligible from the purely numerical viewpoint, is instead essential from the topological viewpoint. In effect, it is so much essential that the opportunity of using an algebraic rather than a differential formulation could be discussed just on the basis of the equality between the left- and right-hand-side terms in Eq. (6). Actually, the $\varepsilon - \delta$ definition of a limit implies choosing an (open) interval, containing the point in which we want to estimate a function, with the aim of making the distance between the points in which we compute the function and the point in which we want to estimate the function as small as we want. In other words, the limit on the left side in Eq. (6) is strictly bonded to the idea of interval of a point and cannot be separated from it. The result of the limit is the value to which the function output appears to approach as the computation point approaches the estimation point. For evaluating this result, we must enough carefully choose the computation points, in order to derive the trend of the output to a specific degree of approximation. That is, the result we obtain by choosing increasingly close points is only an estimation of the actual result and the approximation of the estimation is as much better (the degree of approximation is as much low) as the computation point is close to the estimation point. In conclusion, the $\varepsilon - \delta$ definition of a limit also bounds the limit to the notions of approximation and degree of approximation, or accuracy.

Completely different is the discussion on the right-hand-side function of Eq. (6). Actually, the new function $g(h)$ is computed at a point, the point $h = 0$, without any need of evaluating its trend on an interval. The consequence is that the result we obtain is exact (in a broad sense), and we do not need to prefix any desired accuracy for the result itself. This is very useful from the numerical point of view, but, from the topological point of view, we lose information on what happens approaching the evaluation point. It is the same type of information we lose in passing from the description of a phenomenon in a space to the description of the same phenomenon in the tangent space at the evaluation point.

The idea underlying this paper is that the Cancellation Rule can actually be employed only in those cases where the specific phenomenon uniquely depends on what happens at the point under consideration. In effect, this happens in few physical problems, while, in most cases, the physical phenomenon under consideration also depends on what happens in a neighbourhood centred at the point.

By extension of Eq. (6) to functions of more than one variable, studying the physical phenomenon as if it were a point-wise function means that we are using the right-hand side of Eq. (6), while studying the physical phenomenon as a function of all the points contained in a neighbourhood means that we are using the left-hand side, with h approaching zero but never equal to zero. In the first case, we are facing a differential formulation, while, in the second case, we are facing an algebraic formulation.

Operatively, we are using an algebraic formulation whenever we choose increasingly close points (to both the right and the left) of the estimation point, until the outputs remain constant to one decimal place beyond the desired accuracy for two or three calculations. How much the computation points must be close to the estimation point depends on how fast the result of the limit is approached as we approach the point in which the limit is estimated. Therefore, the dimension of the neighbourhood is fixed by the trend of the phenomenon around the point under consideration, or, in other words, the distance δ for the evaluation of $f'(c)$ depends both on the error ε and on $f''(c)$. The information we lose by using the Cancellation Rule lies just in the trend of the phenomenon, that is, in the curvature, since the curvature cannot be accounted for in passing from a space to its tangent space at the evaluation point.

In the differential formulation, the notion of limit is used not only for defining derivatives, but also densities. In this second case, the denominator that tends to zero has the dimensions of a length raised to the power of 1, 2, or 3. The Cancelation Rule for limits can be employed also in this second case, by factorizing and cancelling length scales in dimension 1, 2, or 3, respectively. This leads to point-wise variables in any cases, the line, surface, and volume densities. Finally, the Cancelation Rule for limits is used also for finding rates, by factorizing and cancelling time scales in dimension 1. This last time, the limit, which is a time derivative, provides an instant-wise variable.

In conclusion, with reference to the space of the physical phenomena, the differential formulation provides the numerical solution in the tangent space of degree 0, where we can describe each physical phenomenon in terms of the space elements of degree 0, the points, and the time elements of degree 0, the time instants. Conversely, the algebraic formulation allows us to take account of, we could say, the curvatures in space and time at a point, where a point of the space of the physical phenomena is a given physical phenomenon, in a given configuration, at a given time instant.

3 A Comparison Between Algebraic Formulation and Non-Standard Analysis

In the previous Section, we have argued that the solution given by the Cancelation Rule for limits is the projection of the actual solution from the multi-dimensional space to the tangent space of degree 0. In fact, the cancelation of the common factors in numerator and denominator acts as a projection operator and the equality in Eq. (4) should more properly be substituted by a symbol of projection. Consequently, the solution of the differential formulation is the shadow of the actual solution in the tangent space of degree 0. On the contrary, the algebraic formulation, by avoiding the projection process, provides us with a higher degree solution, approximated in any case, which is more adherent to the physical nature of the phenomenon under consideration.

By using the language of non-standard analysis, which is a rigorous formalization of calculations with infinitesimals, the infinite and infinitesimal quantities can be treated by the system of hyperreal numbers, or hyperreals, or nonstandard reals. Denoted by ${}^*\mathbb{R}$, the hyperreal numbers are an extension of the real numbers, $\mathbb{R}$, that contains numbers greater than anything of the form:

$$1+1+\ldots+1. \tag{7}$$

Such a number is infinite, and its reciprocal is infinitesimal.

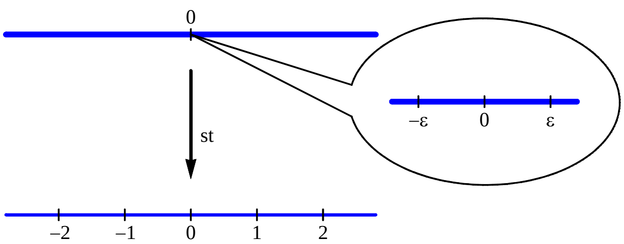

Fig.1: The bottom line represents the “thin” real continuum. The line at top represents the “thick” hyperreal continuum. The “infinitesimal microscope” is used to view an infinitesimal neighborhood of 0.

Non-standard analysis deals primarily with the hyperreal line, which is an extension of the real line, containing infinitesimals, in addition to the reals (Fig. 1). In the hyperreal line every real number has a collection of numbers (called a monad, or halo) of hyperreals infinitely close to it.

The standard part function is a function from the limited (finite) hyperreal to the reals. It associates with a finite hyperreal x, the unique standard real number x_0 which is infinitely close to it (Fig. 1):

$$\mathrm{st}(x) = x_0 . \tag{8}$$

As such, the standard part function is a mathematical implementation of the historical concept of adequality introduced by Pierre de Fermat. It can also be thought of as a mathematical implementation of Leibniz’s Transcendental Law of Homogeneity.

The standard part function was first defined by Abraham Robinson as a key ingredient in defining the concepts of the calculus, such as the derivative and the integral, in non-standard analysis.

The standard part of any infinitesimal is 0. Thus, if N is an infinite hypernatural, then $1/N$ is infinitesimal, and

$$\mathrm{st}\left(\frac{1}{N}\right) = 0 . \tag{9}$$

The standard part function allows the definition of the basic concepts of analysis, such as derivative and integral, in a direct fashion. The derivative of f at a standard real number x becomes

$$f'(x) = \mathrm{st}\left(\frac{{}^*f(x+\Delta x) - {}^*f(x)}{\Delta x}\right), \tag{10}$$

where Δx is an infinitesimal, smaller than any standard positive real, yet greater than zero, and ${}^*f$ is the natural extension of f to the hyperreals ($*$ is the transfer operator applied to f). Similarly, the integral is defined as the standard part of a suitable infinite sum.

In this approach, $f'(x)$ is the real number infinitely close to the hyperreal argument of st. For example, the non-standard computation of the derivative of the function $f(x) = x^2$ provides

$$f'(x) = \mathrm{st}\left(\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right) = \mathrm{st}(2x+\Delta x) = 2x , \tag{11}$$

since

$$2x + \Delta x \approx 2x , \tag{12}$$

where the symbol “$\approx$” is used for indicating the relation “is infinitely close to”. In order to make

$f'(x)$ a real-valued function, we must dispense with the final term, Δx, which is the error term. In the standard approach using only real numbers, that is done by taking the limit as Δx tends to zero. In the non-standard approach using hyperreal numbers, the quantity Δx is taken to be an infinitesimal, a nonzero number that is closer to 0 than to any nonzero real, which is discarded by the standard part function.

The notion of limit can easily be recaptured in terms of the standard part function, st, namely:

$$\lim_{x\to c} f(x) = L, \tag{13}$$

if and only if, whenever the difference $|x-c|$ is infinitesimal, the difference $|f(x)-L|$ is infinitesimal, as well. In formulas

$$\mathrm{st}(x) = c \Rightarrow \mathrm{st}(f(x)) = L. \tag{14}$$

The standard part of x is sometimes referred to as its shadow. Therefore, the derivative of $f(x)$ is the shadow of the difference quotient.

We can thus conclude that the standard part function is a form of projection from hyperreals to reals. As a consequence, using the algebraic formulation is somehow similar to performing non-standard calculus, the modern application of infinitesimals, in the sense of non-standard analysis, to differential and integral calculus. In effect, the extension of the real numbers, $\mathbb{R}$, is equivalent to providing the space of reals with a supplementary structure of infinitesimal lengths. This configures the hyperreal number system as an infinitesimal-enriched continuum, and the algebraic approach can be viewed as the algebraic version of non-standard calculus.

The great advantage of the infinitesimal-enrichment is that of successfully incorporating a large part of the technical difficulties at the foundational level of non-standard calculus. Similarly, in the algebraic formulation many numerical problems, mainly instability or convergence problems, are avoided by the presence of a supplementary structure of (finite) lengths both in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

4 The Truly Algebraic Method: the Cell Method

One of the main consequences of using the left- rather than the right-hand side in Eq. (6) is that the nature of physical variables is different in the algebraic rather than the differential formulation, global in the first case and local in the second case. Broadly speaking, the global variables are variables that are neither densities nor rates of other variables. In particular, we will call:

- Global variable in space, or space global variable, a variable that is not the line, surface, or volume density of another variable.
- Global variable in time, or time global variable, a variable that is not the rate or time derivative of another variable.

The field variables are obtained from the global variables as densities of space global variables and rates of time global variables. Due to their point-wise nature, they are local variables.

In the differential formulation, some variables arise directly as functions of points and time instants, while the remaining variables are reduced to points and time instants functions by performing densities and rates and making use of the Cancelation Rule for limits. Thus, the physical variables of the differential formulation are point-wise and/or instant-wise field functions.

Conversely, by avoiding factorization and cancelation, the algebraic formulation uses global variables. Moreover, since, in doing so, the algebraic formulation preserves the length and time scales of the global physical variables, the physical variables, in spatial description, turn out to be naturally associated with one of the four space elements (point, line, surface, and volume, which are denoted with their initial capital letters in bold, $\mathbf{P}$, $\mathbf{L}$, $\mathbf{S}$, and $\mathbf{V}$, respectively, as shown in Fig. 2) and/or with one of the two time elements (time instant and time interval, which are denoted with $\mathbf{I}$ and $\mathbf{T}$, respectively, as shown in Fig. 3).

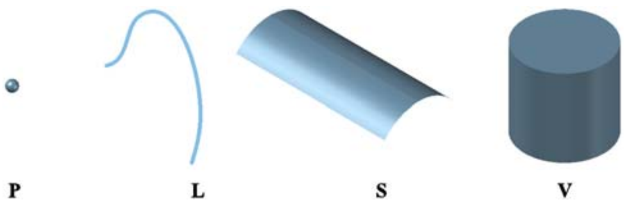

Fig.2: The four space elements and their notations.

Fig.3: The two time elements and their notations.

By using global variables, it is possible to obtain an algebraic formulation directly and, what is most important, the global variables involved in obtaining the formulation do not have to be differentiable functions. This observation has inspired the philosophy of the Cell Method (CM), the truly and unique algebraic method, at the moment. The ability of the CM to solve some of the problems affected by spurious solutions in the differential formulation lies, in part, just on the association between variables and space and/or time elements.

5 Conclusions

The Cancelation Rule for limits, extensively used in the differential formulation, acts on the actual solution of a physical problem as a projection operator that degrades the solution itself. The Cell Method (CM) avoids to use the Cancelation Rule for limits and adopts a formulation that has many contact points with the iterative technique for performing the limit process. This allows the CM to associate any physical variable with the geometrical and topological features, usually neglected by the differential formulation. The governing equations are derived in algebraic manner directly, by means of the global variables.

The direct algebraic approach can be viewed as the algebraic version of non-standard calculus. In fact, the extension of the real numbers with the hyperreal numbers, which is on the basis of non-standard analysis, is equivalent to providing the space of reals with a supplementary structure of infinitesimal lengths. In other words, it is an attempt to recover the loss of length scales due to the use of the Cancelation Rule for limits, in differential formulation. For the same reasons, the CM can be viewed as the numerical algebraic version of those numerical methods that incorporate some length scales in their formulations. This incorporation is usually done, explicitly or implicitly, in order to avoid numerical instabilities. Since the CM does not need to recover the length scales, because the metric notions are preserved at each level of the direct algebraic formulation, the CM is a powerful numerical instrument that can be used to avoid some typical spurious solutions of the differential formulation.

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