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ON NONLOCALITY AND LOCALITY: DIFFERENTIAL AND DISCRETE FORMULATIONS

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ABSTRACT
Nonlocal approaches are discussed with regard to the differential and discrete formulations. Nonlocality was found to be a concept non attaining to the description of the material, but of the physical phenomenon. When using the differential formulation for modeling heterogeneous materials, a length scale must be introduced into the material description. This need has been here justified on the basis of the geometrical information which has been lost in performing the limit process. It was shown how, avoiding the limit process, a length scale is intrinsically taken into account into a discrete formulation. This made it possible to discuss the opportunity of using nonlocality in order to give respectability to strain-softening damage models.

1 INTRODUCTION

One of the main research fields in past years concerns the modeling of heterogeneous materials, for which the use of the classical local continuum concept does not seem to be adequate. This concept leads to constitutive models falling within the category of simple nonpolar materials (Noll [16]), with the stress at a given point uniquely depending on the current values, and possibly also the previous history, of deformation and temperature at that point only (Bažant [2]).

Krumhansl [24], Rogula [27], Eringen [7], Kunin [25], and Kröner [23] promulgated the idea that heterogeneous materials should properly be modeled by some type of nonlocal continuum. Some preliminary ideas on nonlocal elasticity can be traced back to the late 19th century (Duhem [5]). Nonlocal continua are continua in which the stress at a certain point is not a function of the strain at the same point, but a function of the strain distribution over a certain representative volume of the material centered at that point (Bažant [1]). Thus, nonlocality is tantamount to an abandonment of the principle of the local action of classical continuum mechanics (Bažant [2]).

Nonlocal approaches were employed in various branches of physics. In solid mechanics, the need to improve the classical continuum description with an internal length parameter is motivated by the impossibility of modeling the size effect in the context of the classical plasticity.

2 COMPARISON BETWEEN DIFFERENTIAL AND DISCRETE FORMULATIONS

The analysis of solid mechanics is traditionally based on a differential formulation. This formulation requires field functions, which have to depend on point position, x, y, z, and instants, t. Only on this condition is it possible to find the derivatives and, then, to apply the differential formulation. So, if the field functions are not directly described in terms of x, y, z, and t, they are obtained from global variables, by performing densities and rates.

Global variables are domain variables, depending on x, y, z, and t, but also on line extensions, L, areas, S, volumes, V, and time intervals, Δt. Reduction of global variables to point and instant variables is not physically appealing. As far as the point-position reduction of variables is concerned, one should consider that any physical phenomenon occurs in space. Space, with its multi-dimensional geometrical structure, is the natural referent of phenomena. In other words, physics has an intrinsic length scale. Consequently, all global variables are implicitly associated with geometrical objects provided with an extension (points, but also lines, areas and volumes).

On the other hand, it is now a commonly accepted fact that the solution of a problem can be governed by the ratio of the physical dimensions of a structure to an intrinsic material length. The dependence on the size effect cannot be resolved by a differential formulation, since the geometrical information, i.e. the intrinsic length scale of physics, has been lost. According to the mathematical definition of nonlocality given by Rogula [28], the differential formulation could be considered as intrinsically local, since differential operators satisfy the condition of locality. It is here assumed that the intrinsic locality of differential operators is the main reason why nonlocal material models must be introduced in order to satisfy the nonlocality of physical phenomena. In other words, nonlocality attains to physics, and not necessarily to some type of material model. If the problem is studied in the context of the differential formulation, which is a local formulation, nonlocality must be recovered by means of some type of enriched continuum models. Otherwise, if nonlocality is implicit in the formulation, there is no longer any need to employ nonlocal material models for the description of solid mechanics.

As a proof of what has been asserted, one should consider that the theories of nonlocal elasticity advanced by Eringen and Edelen in the early 1970s (Edelen [6], Eringen [8],[9]) attributed a nonlocal character to body forces, mass, entropy, and internal energy. These are all global variables whose geometrical referent is a volume. It is thus clear that they, like all variables whose geometrical referent is more than zero-dimensional, cannot be properly described in a context in which all variables are related to points.

The use of a discrete formulation instead of a differential one is justified by the heterogeneous microstructure of materials. In fact, since matter is discrete on a molecular scale, the density finding process and the notion itself of density lose their physical sense. Moreover, when performing densities and rates, the intention is to formulate the field laws in an exact form. Nevertheless, the differential formulation can only be solved for very simple geometries and particular boundary conditions. To obtain a solution in the general case, the differential equations must be expressed in a discrete form (for each differential method). Consequently, the final solution is an approximation in all cases. It therefore seems unnecessary to use exact equations if, to solve them, we must introduce some kind of approximation.

In order to clarify why physics has an intrinsic length scale, let us now choose a set of points in space, the set of primal nodes $\boldsymbol{P}$ (black points in Fig. 1). Lines connecting primal nodes (black lines) define a spatial mesh, the primal cell complex. Edges, areas, volumes of the primal cell complex are, respectively, the primal sides $\boldsymbol{L}$, surfaces $\boldsymbol{S}$, volumes $\boldsymbol{V}$. Now, consider the surfaces, locus of the points which are equidistant from each pair of primal nodes (gray surfaces). These surfaces define a second spatial mesh (Fig. 1), the dual cell complex. Points, edges, areas, volumes of the dual cell complex are, respectively, the dual nodes $\tilde{\boldsymbol{P}}$, sides $\tilde{\boldsymbol{L}}$, surfaces $\tilde{\boldsymbol{S}}$, volumes $\tilde{\boldsymbol{V}}$. It can be shown (Tonti [29]) that the variables of each physical theory are not related to the geometrical objects of one cell complex only. A relationship between variables and geometrical objects of both cell complexes is established.

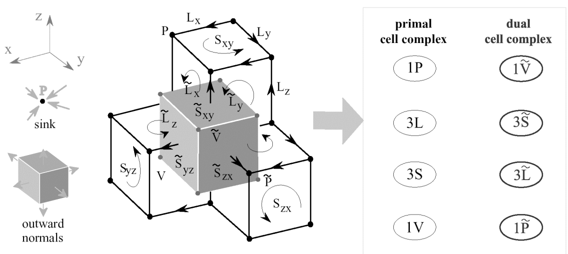

Figure 1: Correspondence between objects of the primal and dual cell complexes in 3D space.

Geometrical and temporal structures of space can be endowed with orientation. As far as the spatial elements are concerned, whenever the orientation of a space element lies on the element itself, an *inner orientation* is established, while, whenever the orientation of a space element depends on the space in which the element is embedded, an *outer orientation* is established. Inner and outer orientations are shown in Fig. 1. It can easily be seen how providing the primal complex with an inner orientation, all elements of the dual complex are endowed with an outer orientation.

In accordance with Hallen [18], Penfield and Haus [26], and Tonti [29], all physical variables belong to one of the following three classes: *configuration variables*, describing the field configuration, *source variables*, describing the field sources, and *energetic variables*, given by the product of a configuration variable for a source variable.

Not only variables, but also their classification can be put in relationship with geometry. The objects of the primal cell complex are natural geometrical referents of configuration variables, while the objects of the dual cell complex are natural geometrical referents of source variables. The consequence of this property, in conjunction with the relationship between cell complexes and type of orientation, is remarkable: by providing the primal cell complex with an inner orientation, configuration variables of any field theory are associated with cells endowed with an inner orientation, while source variables are associated with cells endowed with an outer orientation.

Due to the correspondence between variables and geometry, including duality and orientation properties, a mathematical description of a phenomenon cannot leave out of consideration the geometrical structure of the phenomenon itself. The coordinate systems of the differential formulation are not sufficient for describing phenomena, since they are adequate to describe only points in space and time. A formulation aimed at preserving the geometrical structure of phenomena must use some kind of spatial and temporal elements. In this sense, cell complexes are much more than a domain discretization: they are the generalization of the coordinate systems, when the geometrical counterpart of physical variables is taken into account.

The preservation of the geometrical structure of phenomena is the main intent of the Cell Method (CM), a method developed by Tonti [30], providing a direct finite formulation of field equations, without requiring a differential formulation. This is the reason why the CM uses a complex of primal and dual cells as natural geometrical referent of physical variables. The association of physical variables to elements of a cell complex and its dual was introduced by Okada [17] and Branin [3]. In the CM, the strong coupling between physical variables and oriented space elements becomes the key to give a direct discrete formulation to physical laws of fields. This allows the CM to highlight the geometrical, algebraic and analytical structure which is common to different physical theories, leading to a unified description of physics.

Speaking of geometric content and of nonlocality is the same thing. We can thus state that the CM is a theory intrinsically preserving nonlocality. In conclusion, the discrete formulation is more appealing than the differential formulation from the physical point of view. The CM is also more appealing as far as the discussion on the discrete nature of matter is concerned. Actually, since the use of point functions is no longer needed by leaving the differential formulation, the CM deals with (discrete) equations that are not in conflict with the discrete nature of matter.

4 NONLOCALITY IN STRAIN-SOFTENING MODELING

The enrichment of the classical continuum by incorporating nonlocal effects into the constitutive equations is often used in differential formulations in order to avoid the ill-posedness of boundary value problems with strain-softening constitutive models. When the material tangent stiffness matrix ceases to be positive definite, the governing differential equations may lose ellipticity. FEM solutions of such problems exhibit a pathological sensitivity to the element size and do not converge to physically meaningful solutions as the mesh is refined (Jirásek [20]). Actually, the boundary value problem does not have a unique solution with continuous dependence on the given data (Jirásek [21]). To remedy the loss of ellipticity, a length scale must be incorporated, implicitly or explicitly, into the material description or the formulation of the boundary value problem (Chen [4]). A properly formulated enhancement has a regularizing effect, since it acts as a localization limiter that restores the well-posedness of the boundary value problem. The actual width of the zone of localized plastic strain is related to the heterogeneous material microstructure and can be correctly predicted only by models having a parameter with the dimension of length (Jirásek [22]).

The nonlocal approach arises from the absence of a length scale in the differential formulation, which is a direct consequence of loosing metric notions when performing the limit process. Since physics has an intrinsic length scale, the differential equation arising from the limit process cannot describe physical phenomena properly. Thus, the lost metric notions must be re-entered somehow. Nonlocal approaches re-enter the metric notions by incorporating a length scale into the constitutive equations. This incorporation is not per-se necessary at all. It is required by the formulation. What is necessary is to preserve the nonlocality of phenomena. This may be achieved by preserving the metric information by means of a discrete formulation. It is then possible to use a local constitutive relationship if the numerical simulation is performed by means of the CM.

Transition from highly localized strains to displacement discontinuities embedded in the interior of finite elements can be used to remedy the loss of convergence when body forces are present (Jirásek [19]). As pointed out in Jirásek [20], this approach is appealing from the physical point of view, since in the final stage of the degradation process the material should no longer be considered as a continuum. Nevertheless, it is here argued that this transition corresponds to a description of the stress field in terms of displacements, and not of strains (Fig. 2). Thus, the stress field is not related to the microscopic behavior of the material, but to the macroscopic behavior of the structure. Microscopic and macroscopic behavior may differ when the structure is no longer a continuum (Ferretti [12]). This means that the transition is equivalent to introduce a relationship between effective stress and effective strain, which not necessarily is strain-softening. If this is the case, the existence itself of the strain-softening behavior is not ensured by this approach. In other words, by using the transition is not clear whether a case of strain-softening is actually modeled. Therefore, capturing the correct crack trajectory without any numerical instabilities through a transition technique cannot be considered a proof of the strain-softening existence. Displacement discontinuity with opening of macroscopic cracks has shown itself to be per-se sufficient to model softening branch (Fig. 3) and size effect (Fig. 3) in the load-displacement diagram of compressed specimens, even if a monotone constitutive law is used (Ferretti [11], [15]). Results in Fig. 3 have been provided by means of a CM code with intra-element propagation and automatic remeshing (Ferretti [10]), using a local constitutive law. The coupling between CM and local monotone law is also able to simulate compression tests on concrete cylinders, wrapped with sheets of carbon fiber composites (CFRP, Fig. 4).

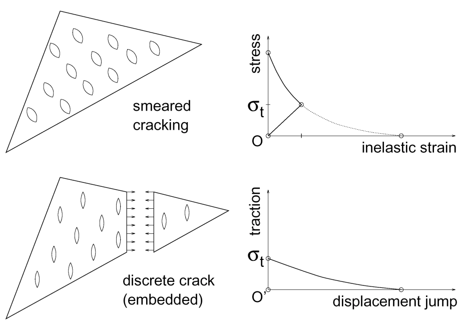

Figure 2: Transition from a continuum model to a discontinuity (after Jirásek [20]).

It may be concluded that softening in load-displacement diagrams attains to the structural response and does not necessarily correspond to material softening, whose existence is not guaranteed at all. The problem of the existence of strain-softening is actually still an open issue. Recently, a new procedure for the identification of concrete local laws has been proposed (Ferretti [13], [15]), showing that a monotone constitutive law, the effective law, is derived if the concrete specimen is not considered as a continuum, according to the experimental evidence (Ferretti [14]).

Nonlocal models with intra-element propagation aiming at simulating the modified interactions between material points, due to cracking, must continuously recompute the interaction weights for all interacting pairs of integration points. Recomputation is needed since long-range interaction between material points becomes more and more difficult, and finally impossible, as the crack propagates. Thus, the interaction length must be decreased. Matters are different with a CM code with intra-element propagation. Actually, since the nonlocal approach is implicit into the CM, the modified nonlocal behavior is automatically taken into account as the geometry is updated.

In Bažant [1] and Jirásek [22], it was shown that numerical instabilities do not occur only if softening laws taking into account both the local and nonlocal effects are used. This means that the principle of the local action of the classical continuum mechanics must somehow be taken into account even in a nonlocal approach. This is exactly what happens in a CM code with a local constitutive model, being nonlocality ensured by the discrete formulation. The use of a local/nonlocal constitutive model in the FEM is thus equivalent to the use of a local constitutive model in the CM. This equivalence is also proved by the capability of the CM with local constitutive model of succeeding where classical plasticity fails, requiring an improvement of the classical continuum description: modeling the size effect (Fig. 3; Ferretti [11]). Thus, one of the main historical reason for improving the classical continuum description fails if the differential formulation is abandoned in favor of a discrete one.

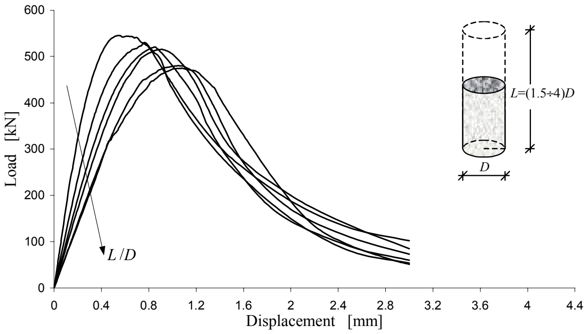

Figure 3: Numerically evaluated size effect on load-displacement diagrams.

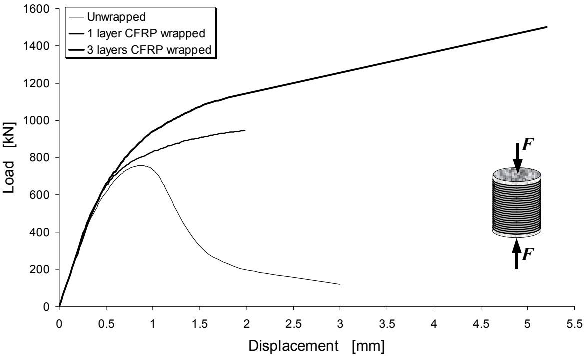

Figure 4: Numerical load-displacement curves for unwrapped and CFRP wrapped specimens.

Finally, nonlocal theories aiming at regularizing the localization problem usually neglect nonlocal elastic effects, and apply nonlocal averaging only to an internal variable (or thermodynamic force) linked to dissipative processes (Jirásek [22]). The implicit nonlocality of the CM also allows automatic estimation of nonlocal effects in the elastic regime. The transition between elastic and strain-localization regimes is no longer critical for the accurateness of the numerical analysis and distinguishing between ante and after strain-localization regime is no longer necessary.

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