IGF 17
XVII National Conference Italian Group of Fracture
Bologna (Italy), June 16-18, 2004

# On Nonlocality and Locality: Differential and Discrete Formulations

Elena Ferretti[1]

**Short abstract:** One of the main research fields in past years concerns the modeling of heterogeneous materials. For these materials, the use of the classical local continuum concept does not seem to be adequate. The classical local continuum concept leads to constitutive models falling within the category of simple nonpolar materials (Noll 1972). For these materials, the stress at a given point uniquely depends on the current values, and possibly also the previous history, of deformation and temperature at that point only (Bažant and Jirásek 2002).
Beginning with Krumhansl (1965), Rogula (1965), Eringen (1966), Kunin (1966), and Kröner (1968), the idea was promulgated that heterogeneous materials should properly be modeled by some type of nonlocal continuum. Some preliminary ideas on nonlocal elasticity can be traced back to the late 19th century (Duhem 1893). Nonlocal continua are continua in which the stress at a certain point is not a function of the strain at the same point, but a function of the strain distribution over a certain representative volume of the material centered at that point (Bažant and Chang 1984). Thus, nonlocality is tantamount to an abandonment of the principle of the local action of classical continuum mechanics (Bažant and Jirásek 2002).
Local constitutive relations between stress and strain tensors are not adequate for describing the mechanical behavior of solids in the classical differential formulation, since no material is an ideal continuum, decomposable into a set of infinitesimal material volumes, each of which can be described independently. All materials, natural and man-made, are characterized by microstructural details whose size ranges over many order of magnitude (Bažant and Jirásek 2002). In constructing a material model, one must select a certain resolution level below which the microstructural details are not explicitly visible. Instead of refining the explicit resolution level, it is often more effective to use various forms of generalized continuum formulation, dealing with material that are nonsimple or polar, or both. A list of enriched continuum models is provided in Bažant and Jirásek (2002). Among these, a great variety of nonlocal models was developed.
The aim of the present study is to show that nonlocal constitutive relations between stress and strain tensors are not strictly needed to construct a material model. They are required only if a differential formulation is used for modeling nonlocality, since differential operators are local. The physical well-posedness of nonlocality is discussed with regard to the differential and discrete formulations. Nonlocality was found to be a concept not attaining to the description of the material, but of the phenomenon. This made it possible to discuss the opportunity of using nonlocality in order to give respectability to strain-softening damage models. The mathematical and physical well-posedness and the existence of strain-softening are also discussed. When using the differential formulation, a length scale must be introduced into the material description of a strain-softening modeling. This need has been here justified on the basis of the geometrical information which has been lost in performing the limit process. It was shown how, avoiding the limit process, a length scale is intrinsically taken into account into a discrete formulation. Thus, the discrete formulation turns out to be more appealing than the differential formulation with nonlocal approach, from the physical point of view.

[1] Research Assistant, DISTART – Scienza delle Costruzioni, University of Bologna – Alma Mater Studiorum, Bologna, ITALY, elena.ferretti@mail.ing.unibo.it, tel. +39 051 209 3493, fax. +39 051 209 3495.

# On Nonlocality and Locality: Differential and Discrete Formulations

Elena Ferretti[1]

**Extended abstract:** One of the main research fields in past years concerns the modeling of heterogeneous materials. For these materials, the use of the classical local continuum concept does not seem to be adequate. This concept leads to constitutive models falling within the category of simple nonpolar materials (Noll 1972), with the stress at a given point uniquely depending on the current values, and possibly also the previous history, of deformation and temperature at that point only (Bažant and Jirásek 2002).
Beginning with Krumhansl (1965), Rogula (1965), Eringen (1966), Kunin (1966), and Kröner (1968), the idea was promulgated that heterogeneous materials should properly be modeled by some type of nonlocal continuum. Some preliminary ideas on nonlocal elasticity can be traced back to the late 19th century (Duhem 1893). Nonlocal continua are continua in which the stress at a certain point is not a function of the strain at the same point, but a function of the strain distribution over a certain representative volume of the material centered at that point (Bažant and Chang 1984). Thus, nonlocality is tantamount to an abandonment of the principle of the local action of classical continuum mechanics (Bažant and Jirásek 2002).
Local constitutive relations between stress and strain tensors are not adequate for describing the mechanical behavior of solids in the classical differential formulation, since no material is an ideal continuum, decomposable into a set of infinitesimal material volumes, each of which can be described independently. All materials, natural and man-made, are characterized by microstructural details whose size ranges over many order of magnitude (Bažant and Jirásek 2002). In constructing a material model, one must select a certain resolution level below which the microstructural details are not explicitly visible. Instead of refining the explicit resolution level, it is often more effective to use various forms of generalized continuum formulation, dealing with material that are nonsimple or polar, or both. A list of enriched continuum models is provided in Bažant and Jirásek (2002). Among these, a great variety of nonlocal models was developed.
Nonlocal approaches were employed in various branches of physics. In solid mechanics, there exist two types of problem motivated by the need to improve the classical continuum description with an internal length parameter: those with strain-softening and those with no strain-softening at all. They all share the common need of modeling the size effect, which is impossible in the context of the classical plasticity.

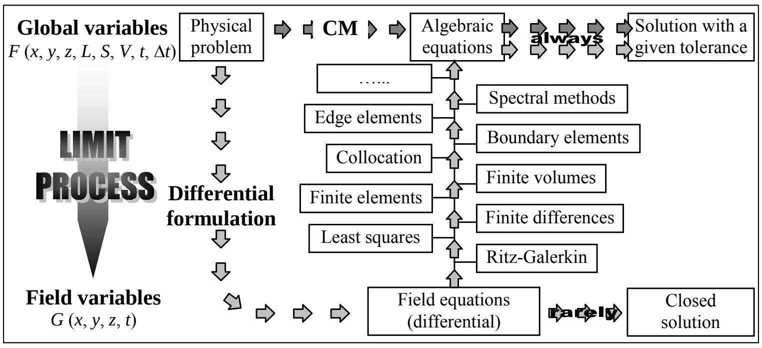


**Fig. 1.** How to achieve the solution thought the Cell Method and the differential formulation

The analysis of solid mechanics is traditionally based on a differential formulation. This formulation requires field functions (Fig. 1), which have to depend on point position, $x$, $y$, $z$, and instants, $t$. Only on this condition is it possible to find the derivatives and, then, to apply the differential formulation. So, if the field functions are not directly described in terms of $x$, $y$, $z$, and $t$, they are obtained from global variables, by performing densities and rates (Fig. 1). Global variables are domain variables, depending on $x$, $y$, $z$, and $t$, but also on line extensions, $L$, areas, $S$, volumes, $V$, and time intervals, $\Delta t$.
Reduction of global variables to point and instant variables is not physically appealing. As far as the point-position reduction of variables is concerned, one should consider that any physical phenomenon occurs in space. Space, with its multi-dimensional geometrical structure, is the natural referent of any phenomenon. In other words, physics has an intrinsic length scale. Consequently, all global variables are implicitly associated with geometrical objects provided with an extension (points, but also lines, areas and volumes). In order to

[1] Research Assistant, DISTART – Scienza delle Costruzioni, University of Bologna – Alma Mater Studiorum, Bologna, ITALY, elena.ferretti@mail.ing.unibo.it, tel. +39 051 209 3493, fax. +39 051 209 3495.

preserve the physical well-posedness of the solving equations, the strict relationship between physics and geometry must be preserved. The preservation is not guaranteed by performing the limit process.

On the other hand, it is now a commonly accepted fact that the solution of a problem can be governed by the ratio of the physical dimensions of a structure to an intrinsic material length. The dependence on the size effect cannot be resolved by a differential formulation, since the geometrical information, i.e. the intrinsic length scale of physics, has been lost. The material lengths enclosed in various forms of generalized continuum theories arise from the homogenization procedure and have their origin in the characteristics of the heterogeneous microstructure that are not explicitly resolved by the differential formulation. In this sense, the differential formulation could be considered as intrinsically local. It is here assumed that the intrinsic locality of differential operators is the main reason why nonlocal material models must be introduced in order to satisfy the nonlocality of physical phenomena. In other words, nonlocality attains to physics, and not necessarily to some type of material model. If the problem is studied in the context of the differential formulation, which is a local formulation, nonlocality must be recovered by means of some type of enriched continuum models. Otherwise, if nonlocality is implicit in the formulation, there is no longer any need to employ nonlocal material models for the description of solid mechanics.

As a proof of what has been asserted, one should consider that the theories of nonlocal elasticity advanced by Eringen and Edelen in the early 1970s (Edelen et al. 1971; Eringen 1972; Eringen and Edelen 1972) attributed a nonlocal character to body forces, mass, entropy, and internal energy. These are all global variables whose geometrical referent is a volume. It is thus clear that they, like all variables whose geometrical referent is more than zero-dimensional, cannot be properly described in a context in which all variables are related to points. Performing the limit process acts as a projection from 3D physics into 0D physics. A description of phenomena living in more-than-zero-dimensional physics is not possible in 0D physics if a length scale is not supplied. On the other hand, if the differential formulation is abandoned in favor of a discrete one, the limit process is no longer performed and we can directly operate in 3D physics. In this latter case, the length scale is naturally associated with global variables and nonlocal effects are intrinsically taken into account.

The use of a discrete formulation instead of a differential one is justified just by the heterogeneous microstructure of materials. In fact, since matter is discrete on a molecular scale, the density finding process and the notion itself of density lose their physical sense. Moreover, when performing densities and rates, the intention is to formulate the field laws in an exact form. Nevertheless, the differential formulation can only be solved for very simple geometries and particular boundary conditions. To obtain a solution in the general case, the differential equations must be expressed in a discrete form (for each differential method, Fig. 1). Consequently, the final solution is an approximation in all cases. It therefore seems unnecessary to use exact equations if, to solve them, we must introduce some kind of approximation.

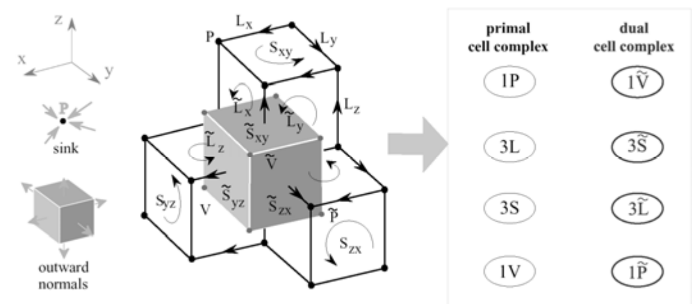


**Fig. 2.** Correspondence between objects of the primal and dual cell complexes in 3D space

In order to clarify why physics has an intrinsic length scale, let us now choose a set of points in space, said the set of primal nodes $\boldsymbol{P}$ (black points in Fig. 2). The lines connecting the primal nodes (black lines in Fig. 2) define a spatial mesh, said the primal cell complex. Edges, areas, and volumes of the primal cell complex are, respectively, the primal sides $\boldsymbol{L}$, surfaces $\boldsymbol{S}$, and volumes $\boldsymbol{V}$. Now, consider the surfaces, locus of the points which are equidistant from each pair of primal nodes (gray surfaces in Fig. 2). These surfaces can be used for building a second spatial mesh (Fig. 2), said the dual cell complex. Points, edges, areas, and volumes of the dual cell complex are, respectively, the dual nodes $\tilde{\boldsymbol{P}}$, sides $\tilde{\boldsymbol{L}}$, surfaces $\tilde{\boldsymbol{S}}$, and volumes $\tilde{\boldsymbol{V}}$.

It can be shown (Tonti 1972) that the variables of each physical theory are not related to the geometrical objects of one cell complex only. A relationship between variables and geometrical objects of both cell complexes is established.

Geometrical and temporal structures of space can be endowed with orientation. As far as the spatial elements are concerned, whenever the orientation of a space element lies on the element itself, an *inner orientation* is established, while, whenever the orientation of a space element depends on the space in which the element is embedded, an *outer orientation* is established. Inner and outer orientations for one dual cell and some primal

cells are shown in Fig. 2. It can easily be seen how providing the primal complex with an inner orientation, all elements of the dual complex are endowed with an outer orientation.
In accordance with Hallen (1962), Penfield and Haus (1967), and Tonti (1972), all physical variables belong to one of the following three classes:

1. *Configuration variables*, describing the field configuration (displacements for solid mechanics, spatial velocity for fluidodynamics, electric potential for electrostatics, temperature for thermal conduction, etc.). All variables linked to configuration variables by operations of sum, integration, difference, division by a length, an area, a volume, and an interval, limit process, time and space derivatives are configuration variables.
2. *Source variables*, describing the field sources (forces for solid mechanics and fluidodynamics, masses for geodesy, electric charges for electrostatics, electric currents for magnetostatics, heat sources for thermal conduction, etc.). All variables linked to source variables by operations of sum, integration, difference, product and division by a length, an area, a volume, and an interval, time and space derivatives are source variables.
3. *Energetic variables*, resulting from the product of a configuration variable for a source variable.

It is worth noting that not only variables, but also their classification can be put in relationship with geometry. The objects of the primal cell complex are natural geometrical referents of configuration variables, while the objects of the dual cell complex are natural geometrical referents of source variables. The consequence of this geometrical property, in conjunction with the relationship between cell complexes and type of orientation, is remarkable: by providing the primal cell complex with an inner orientation, configuration variables of any field theory are associated with cells endowed with an inner orientation, while source variables are associated with cells endowed with an outer orientation.
Due to the correspondence between variables and geometry, including duality and orientation properties, a mathematical description of a phenomenon cannot leave out of consideration the geometrical structure of the phenomenon itself. The coordinate systems of the differential formulation are not sufficient for describing phenomena, since they are adequate to describe only points in space and time. A formulation aimed at preserving the geometrical structure of phenomena must use some kind of spatial and temporal elements.
The preservation of the geometrical structure of phenomena is the main intent of the Cell Method (CM), a method developed by Tonti (2001), providing a direct finite formulation of field equations, without requiring a differential formulation (Fig. 1). This is the reason why the CM uses a complex of primal and dual cells as natural geometrical referent of physical variables. The theory arising from this choice implicitly incorporates a length scale. Speaking of geometric content and of nonlocality is substantially the same thing. We can thus state that the CM is a theory intrinsically preserving nonlocality.
In conclusion, the discrete formulation is more appealing than the differential formulation from the physical point of view. The CM is also more appealing as far as the discussion on the discrete nature of matter is concerned. Actually, since the use of point functions is no longer needed by leaving the differential formulation, the CM deals with (discrete) equations that are not in conflict with the discrete nature of matter.
As previously discussed, a system of dual cells seems to be quite adequate to describe phenomena by preserving the geometrical structure of all involved variables. Cell complexes are much more than a domain discretization: they are the generalization of the coordinate systems, when the geometrical counterpart of physical variables is taken into account. The association of physical variables to elements of a cell complex and its dual was introduced by Okada and Onodera (1951) and Branin (1966). In the CM, the strong coupling between physical variables and oriented space elements becomes the key to give a direct discrete formulation to physical laws of fields. This allows the CM to highlight the geometrical, algebraic and analytical structure which is common to different physical theories, leading to a unified description of physics.
As pointed out by Chen et al. (2000), a meshfree approximation of the Finite Elements Method (FEM) possess intrinsic nonlocal properties, since the approximation functions are not locally constructed. Nonlocal properties of meshfree approximations are exploited to incorporate an intrinsic length scale which regularizes problems with material instabilities. In the CM, a meshfree approach does not directly involve increasing the CM intrinsic degree of nonlocality. Actually, only the procedure of mesh building has changed, and not the approximations used to achieve the solution. This happens since the CM is very ductile and can be easily adapted to a meshfree formulation without having to change the structure of the method. Nevertheless, an increased nonlocality degree can occur if the local mesh building leads to local primal meshes which overlap.
The enrichment of the classical continuum by incorporating nonlocal effects into the constitutive equations is often used in differential formulations in order to avoid the ill-posedness of boundary value problems with

strain-softening constitutive models. When the material tangent stiffness matrix ceases to be positive definite, the governing differential equations may lose ellipticity. FEM solutions of such problems exhibit a pathological sensitivity to the element size and do not converge to physically meaningful solutions as the mesh is refined (Jirásek 1999). Actually, the boundary value problem does not have a unique solution with continuous dependence on the given data (Jirásek and Bažant 2001). To remedy the loss of ellipticity, a length scale must be incorporated, implicitly or explicitly, into the material description or the formulation of the boundary value problem (Chen et al. 2000). A properly formulated enhancement has a regularizing effect, since it acts as a localization limiter that restores the well-posedness of the boundary value problem. This happens since the actual width of the zone of localized plastic strain is related to the heterogeneous material microstructure and can be correctly predicted only by models having a parameter with the dimension of length (Jirásek and Rolshoven 2002).

Incorporating nonlocal effects into the constitutive equations is required by the use of the differential formulation, which is intrinsically local. If the goal is to perform a nonlocal analysis by means of the differential formulation, the constitutive equations must necessarily be modified in order to incorporate nonlocal effects. If the goal is to perform a nonlocal analysis by means of the CM, no need to modify the constitutive equations arises, since the CM is intrinsically nonlocal. The question is not merely which type of continuum to associate with a differential or discrete formulation, nonlocal or local, respectively. The discussion on nonlocality and locality with reference to the differential and discrete formulation takes on a deeper meaning. As previously stated, the nonlocal approach arises from the absence of a length scale in the differential formulation. This is a direct consequence of loosing metric notions when performing the limit process. Since physics has an intrinsic length scale, the differential equation arising from the limit process cannot describe physical phenomena properly. Thus, the lost metric notions must be re-entered somehow. Nonlocal approaches re-enter the metric notions by incorporating a length scale into the constitutive equations. This incorporation is not per-se necessary at all. It is required by the formulation. What is necessary is to preserve the nonlocality of phenomena. This may be achieved by preserving the metric information by means of a discrete formulation. It is then possible to use a local constitutive relationship if the numerical simulation is performed by means of the CM.

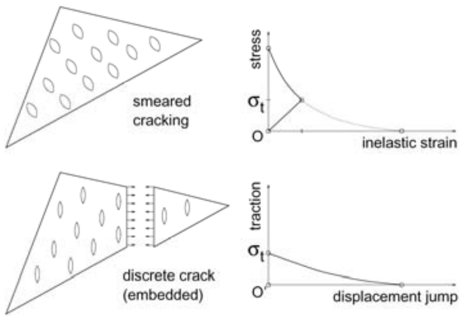


**Fig. 3.** Transition from a continuum model to a discontinuity (after Jirásek 1999)

Most nonlocal damage formulations lead to a progressive shrinking of the zone in which local strains increase (Pijaudier-Cabot and Bažant 1987; Jirásek and Zimmermann 1998). The thickness of the zone of increasing damage can never be smaller than the support diameter of the nonlocal weight function. Numerical problems thus occur, when the residual stiffness of the material inside this zone becomes too small. These numerical problems are all the more severe if body forces are present, leading to divergence of the equilibrium iteration process. Transition from highly localized strains to displacement discontinuities embedded in the interior of finite elements can be used to remedy the loss of convergence when body forces are present (Jirásek 1998). As pointed out in Jirásek (1999), this approach is appealing from the physical point of view, since in the final stage of the degradation process the material should no longer be considered as a continuum. Nevertheless, it is here argued that this transition corresponds to a description of the stress field in terms of displacements, and not of strains (Fig. 3). Thus, the stress field is not related to the microscopic behavior of the material, but to the macroscopic behavior of the structure. Microscopic and macroscopic behavior may differ when the structure is no longer a continuum (Ferretti 2004a). This means that the transition is equivalent to introduce a relationship between effective stress and effective strain, which not necessarily is strain-softening. If this is the case, the existence itself of the strain-softening behavior is not ensured by this approach. In other words, by using the transition

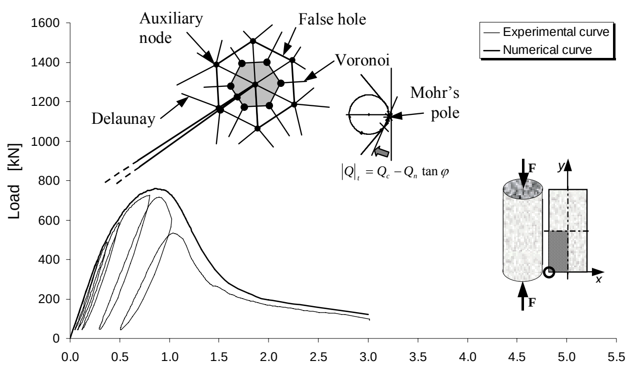


**Fig. 4.** Comparison between numerical and experimental results for compressed specimens

is not clear whether a case of strain-softening is actually modeled. Therefore, capturing the correct crack trajectory without any numerical instabilities through a transition technique cannot be considered a proof of the strain-softening existence. Displacement discontinuity with opening of macroscopic cracks has shown itself to be per-se sufficient to model softening branch (Fig. 4) and size effect (Fig. 5) in the load-displacement diagram of compressed specimens, even if a monotone constitutive law is used (Ferretti 2003b; Ferretti and Di Leo 2003). These results have been provided by means of a CM code with intra-element propagation and automatic remeshing developed by Ferretti (2003a), which uses the local constitutive law shown in Fig. 6. As shown in Fig. 7, the coupling between CM and local monotone law is also able to simulate compression tests on concrete cylinders, wrapped with sheets of carbon fiber composites (CFRP). The good agreement between numerical and experimental results on wrapped concrete is shown in Ferretti and Di Leo (2003).

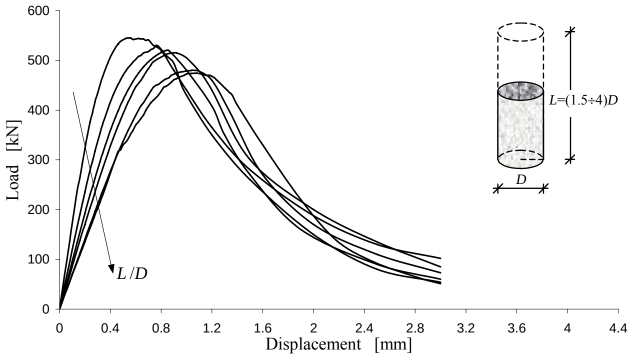


**Fig. 5.** Numerically evaluated size effect on load-displacement diagrams

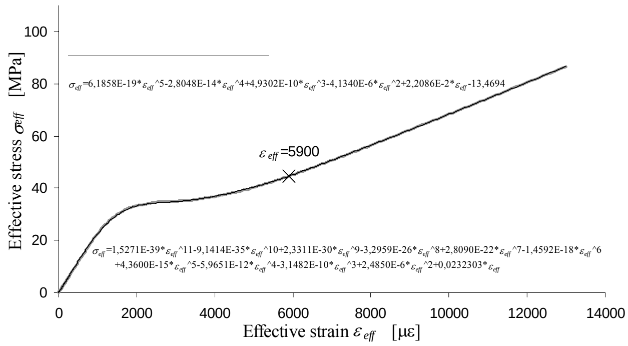


**Fig. 6**. Monoaxial constitutive law adopted for concrete modeling

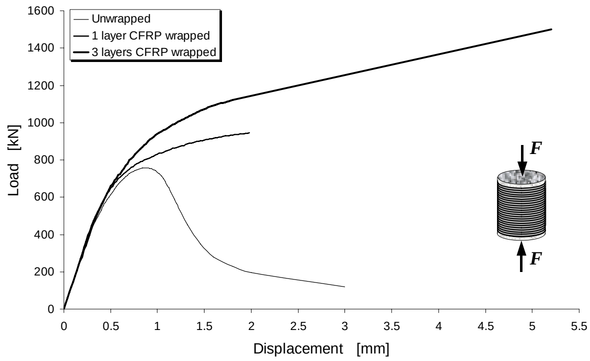


**Fig. 7.** Numerical load-displacement curves for unwrapped and CFRP wrapped specimens

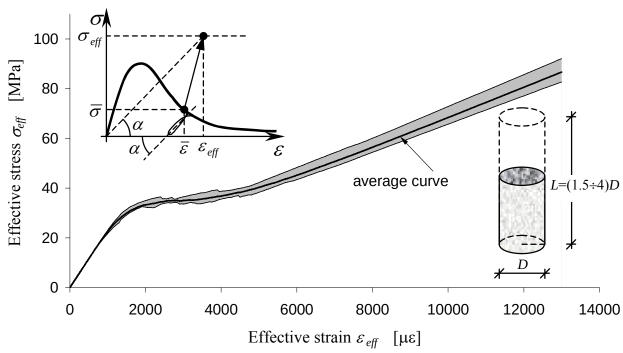


**Fig. 8.** Dispersion range of the effective properties for variable $L/D$ ratio and average curve

It may be concluded that the softening behavior in load-displacement diagrams attains to the structural response and does not necessarily correspond to material softening, whose existence is not guaranteed at all. The problem of the existence of strain-softening is actually still an open issue. Recently, a new procedure for the identification of the local law of concrete has been proposed (Ferretti 2004b; Ferretti and Di Leo 2003), showing that a monotone constitutive law, the effective law, is derived (Fig. 8) if the concrete specimen is not considered as a continuum, according to the experimental evidence (Ferretti 2004c).

A nonlocal model with intra-element propagation aiming at simulating the modified interactions between material points, due to cracking, must be able to continuously recompute the interaction weights for all interacting pairs of integration points. Recomputation is needed since long-range interaction between material points becomes more and more difficult, and finally impossible, as the crack propagates. Thus, the interaction length must be decreased as the crack propagates, with high computational burdens. Matters would be different if a CM code with intra-element propagation were used (Ferretti 2003a). Actually, since the nonlocal approach is implicit into the CM, the modified nonlocal behavior is automatically taken into account as the geometry is updated. No further computation is required when an internal point becomes a boundary node, due to cracking.

In Bažant and Chang (1984) and Jirásek and Rolshoven (2002), it was shown that numerical instabilities do not occur only if softening laws taking into account both the local and nonlocal effects are used. This means that the principle of the local action of the classical continuum mechanics must somehow be taken into account even in a nonlocal approach. This is exactly what happens in a CM code with a local constitutive model, being nonlocality ensured by the discrete formulation. The use of a local/nonlocal constitutive model in the FEM is thus equivalent to the use of a local constitutive model in the CM. This equivalence is also proved by the capability of the CM with local constitutive model of succeeding where classical plasticity fails, requiring an improvement of the classical continuum description: modeling the size effect (Fig. 5; Ferretti, 2003b). Thus, one of the main historical reason for improving the classical continuum description fails if the differential formulation is abandoned in favor of a discrete formulation.

It must finally be noticed that nonlocal theories aiming at regularizing the localization problem usually neglect nonlocal elastic effects, and apply nonlocal averaging only to an internal variable (or thermodynamic force) linked to dissipative processes (Jirásek and Rolshoven 2002). This choice is justified on the basis of the smooth strain distribution characterizing the elastic regime, leading to a good approximation provided by the standard local theory. The implicit nonlocal approach of the CM also allows us to take into account nonlocal effects in the elastic regime automatically. This occurs since the CM nonlocality is derived from geometrical properties naturally linked to physical variables, and not from dissipative processes. Thus, the transition between elastic and strain-localization regimes is no longer critical for the accurateness of the numerical analysis. Distinguishing between ante and after strain-localization regime is no longer necessary.

## References


Bažant, Z. P., and Chang, T. P. (1984). "Is Strain-Softening Mathematically Admissible?" *Proc., 5th Engineering Mechanics Division*, 2, 1377–1380.

Bažant, Z. P., and Jirásek, M. (2002). "Nonlocal Integral Formulations of Plasticity and Damage: Survey of Progress." *J. Eng. Mech.*, 128(11), 1119–1149.

Branin, F. H. (Jr) (1966). "The Algebraic Topological Basis for Network Analogies and the Vector Calculus." *Proc.*, *Symp. on Generalized Networks*, Brooklyn Polit., 453–487.

Chen, J.–S., Wu, C.–T., and Belytschko, T. (2000). "Regularization of Material Instabilities by Meshfree Approximations with Intrinsic Length Scales." *Int. J. Numer. Meth. Engng.*, 47, 1303–1322.

Duhem, P. (1893). "Le Potentiel Thermodynamique et la Pression Hydrostatique." *Ann. Sci. Ecole Norm. S.*, 10, 183–230.

Edelen, D. G. B., Green, A. E., and Laws, N. (1971). "Nonlocal Continuum Mechanics." *Arch. Ration. Mech. Anal.*, 43, 36–44.

Eringen, A. C. (1966). "A Unified Theory of Thermomechanical Materials." *Int. J Eng. Sci.*, 4, 179–202.

Eringen, A. C. (1972). "Linear Theory of Nonlocal Elasticity and Dispersion of Plane Waves." *Int. J Eng. Sci.*, 10, 425–435.

Eringen, A. C., and Edelen, D. G. B. (1972). "On Nonlocal Elasticity." *Int. J Eng. Sci.*, 10, 233–248.

Ferretti, E. (2003a). "Crack Propagation Modeling by Remeshing using the Cell Method (CM)." *CMES*, 4(1), 51–72.

Ferretti, E. (2003b). "Softening and Size Effect CM-Modeling through a Concrete Monotone Law." *J. Eng. Mech. – ASCE*, submitted.

Ferretti, E. (2004a). "A Discussion of Strain-Softening in Concrete", *Int. J. Frac.* (Letters section), 126(1), 3-10.

Ferretti, E. (2004b). "Experimental Procedure for Verifying Strain-Softening in Concrete", *Int. J. Frac.* (Letters section), in press.

Ferretti, E. (2004c). "On Poisson's Ratio and Volumetric Strain in Concrete", *Int. J. Frac.* (Letters section), in press.

Ferretti, E., and Di Leo, A. (2003). "Modelling of Compressive Tests on FRP Wrapped Concrete Cylinders through a Novel Triaxial Concrete Constitutive Law." *SITA*, 5, 20–43.

Noll, W. (1972). "A new Mathematical Theory of Simple Materials." *Arch. Ration. Mech. Anal.*, 48, 1–50.

Okada, S., and Onodera, R. (1951). "Algebraification of Field Laws of Physics by Poincar Process." *Bull. of Yamagata University – Natural Sciences*, 1(4), 79–86.

Hallen, E. (1962). *Electromagnetic Theory*, Chapman & Hall.

Jirásek, M. (1998). "Embedded Crack Models for Concrete Fracture." In: *Computational Modelling of Concrete Structures*, R. de Borst, N. Bićanić, H. Mang, and G. Meschke eds., Balkema, Rotterdam 291–300.

Jirásek, M. (1999). "Computational Aspects of Nonlocal Models." *Proc.*, *ECCM '99*, München, Germany, 1–10.

Jirásek, M., and Bažant, Z. P. (2001). *Inelastic Analysis of Structures*, John Wiley and Sons.

Jirásek, M., and Rolshoven, S. (2002). "Comparison of Integral-Type Nonlocal Plasticity Models for Strain-Softening Materials." *Preprint submitted to Elsevier Science*, 1–63.

Jirásek, M., and Zimmermann, T. (1998). "Rotating Crack Model with Transition to Scalar Damage." *J. Eng. Mech.*, 124(3), 277–284.

Kröner, E. (1968). "Elasticity Theory of Materials with Long-Range Cohesive Forces." *Int. J. Solids Struct.*, 3, 731–742.

Krumhansl, J. A. (1965). "Generalized Continuum Field Representation for Lattice Vibrations." in *Lattice dynamics*, R. F. Wallis ed., Pergamon, London, 627–634.

Kunin, I. A. (1966). "Theory of Elasticity with Spatial Dispersion." *Prikl. Mat. Mekh.* (in Russian), 30, 866.

Penfield, P., and Haus, H. (1967). *Electrodynamics of Moving Media*, M.I.T. Press.

Pijaudier-Cabot, G., and Bažant, Z. P. (1987). "Nonlocal Damage Theory." *J. Eng. Mech.*, 113, 1512–1533.

Rogula, D. (1965). "Influence of Spatial Acoustic Dispersion on Dynamical Properties of Dislocations. I." *Bulletin de l'Académie Polonaise des Sciences*, *Séries des Sciences Techniques*, 13, 337–343.

Tonti, E. (1972). "On the Mathematical Structure of a Large Class of Physical Theories." *Rend. Acc. Lincei*, 52, 48–56.

Tonti, E. (2001). "A Direct Discrete Formulation of Field Laws: the Cell Method." *CMES*, 2(2), 237–258.