International Journal of Fracture **126:** L3–L10, 2004.

A DISCUSSION OF STRAIN-SOFTENING IN CONCRETE

Elena Ferretti
Alma Mater Studiorum – University of Bologna, Faculty of Engineering, DISTART – Scienza delle Costruzioni, Viale Risorgimento 2, 40136 Bologna, Italy
E-mail: elena.ferretti@mail.ing.unibo.it

Abstract. The question of whether concrete strain-softening is a real material property is discussed here. The discussion is given from both the physical and the analytical point of view. New evaluations of the actual nature of strain-softening are added to those existing in literature.

1. Introduction. The strain-softening of a material is the decline of stress at increasing strain. Strain-softening diagrams are obtained from displacement-controlled compression tests on concrete-like materials. In 1903, Hadamard considered strain-softening as an unacceptable feature for a constitutive equation, since the compressive wavespeed, *V*, function of the tangent modulus *E* and the Poisson's ratio υ (1), ceases to be real if *E* becomes negative, as in strain-softening branches:

$$V = \sqrt{\frac{E}{\rho}\frac{(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}}\,. \tag{1}$$

Even if questionable under certain viewpoints, Hadamard's observation gave rise to the question of whether strain-softening really exists or not. Since then, strain-softening has been widely regarded as inadmissible by several authors (Bažant, 1984). The problem of strain-softening in continuum dynamics has been intensely discussed in regard to large-scale finite element computations (Bažant and Belytschko, 1985). It has been questioned (Bergan, 1983; Hegemier and Read, 1983; Sandler and Wright, 1983) whether strain-softening in a continuum is a sound concept from the mathematical point of view. The question was, whether or not strain-softening is a real material property or merely the result of inhomogeneous deformation caused by experimental techniques. It was found (Sandler and Wright, 1983) that the standard approach interpreting load-displacement experimental curves with softening as stress-strain curves does not lead to a meaningful representation of dynamic continuum problems in a physical and mathematical manner. Actually, the stability in the sense of Hadamard (1903), i.e., proper-posedness, is not satisfied, since in the softening regime the governing equation are elliptic instead of hyperbolic. A review of the most relevant studies on strain-softening existence is given in Ferretti and Di Leo (2003). In particular, Hudson et al. (1971) suggested that softening is not a material property, but is essentially due to scaling the applied force by the original cross-sectional area rather than the actual cross-sectional area. Also the other studies mentioned here share the common idea of the non constitutive nature of strain-softening. Nevertheless, they were not able to provide an identifying procedure from the experimental data to a monotonic constitutive law for concrete. They treated the problem from the theoretical viewpoint only, since it was estimated (Hegemier and Read, 1983) to be extremely difficult, if not impossible, to track the effective cross-sectional area experimentally at each stage of the failure process. The impossibility of achieving a new constitutive proposal is the main reason for which this field of research rapidly fell out of favor. The aim of the present work is to further investigate the actual nature of strain-softening, from the physical and the analytical points of view. A suggestion on constitutive relations is also given by Ferretti (2001), based on a new analysis of experimental data.

2. Strain-softening existence: physical point of view. In order to derive a constitutive law in uniaxial compression from experimental data, it is a common practice to define the average stress $\bar{\sigma}$ and the average strain $\bar{\varepsilon}$ as shown in Fig. 1, and to assume a uniform state of stress and strain in the specimen. With this assumption, the average stress and strain are equal, respectively, to the stress and strain at a generic point, σ and ε (Fig. 1). Evaluating the average quantities is thus considered equivalent to performing the limit process of the corresponding difference quotients. In this case, the σ-ε relationship in Fig. 1 represents the uniaxial constitutive law for monotonic strain processes. This relationship is considered to be representative of the mechanical behavior of the material.

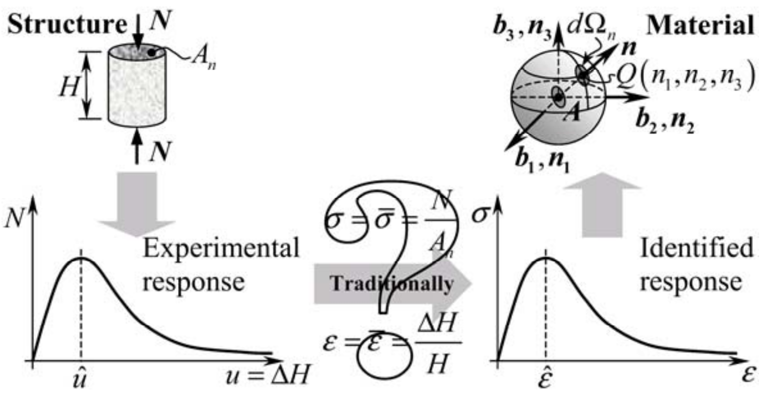

Figure 1. Traditional identification of uniaxial constitutive law by experimental tests.

However, one can make the following remarks:

1. The σ-ε law in Fig. 1 is size- and shape-effect sensitive, while a constitutive law should not exhibit either a size-, or a shape-effect.
2. The identification procedure in Fig. 1 consists of a mere change of scale. Thus, *N*-*u* and σ-ε curves are homothetic, that is, identical in shape. In particular, they both exhibit a softening behavior. The softening behavior in

the *N*-*u* relationship has a well known physical meaning, linked to the structure instability. Since any structural property is no longer definable when the structural dimensions tend to zero, it is not clear what a softening branch could represent in the curve of the material response (σ-ε curve), which is defined in the point. In other words, the concept of structural instability is associated with a certain body configuration. Extrapolation of this concept to a point property is not formally correct, since a configuration is not definable for the point. If some type of instability arises in the point, it would be not necessarily associated with strain-softening. On the other hand, the lack of stress uniqueness involved by strain-softening does not seem to be reasonable for a law describing a constitutive behavior. A biunique relationship between stress and strain is expected in such a type of relationship.

These inconsistencies come from the impossibility of performing mechanical tests directly on the material (Ferretti and Carli, 1999): the object of testing is never the material, but a specimen, that is to say, a structure interacting with the test-machine (Fig. 1). Thus, experimental results (*N*-*u*) characterize the behavior of the specimen-test machine system, while they are not at all representative of the constitutive behavior of the material. In particular, the softening branch has a meaning that is only linked to the structural instability. This branch cannot provide information on the material behavior, but through an identifying model.

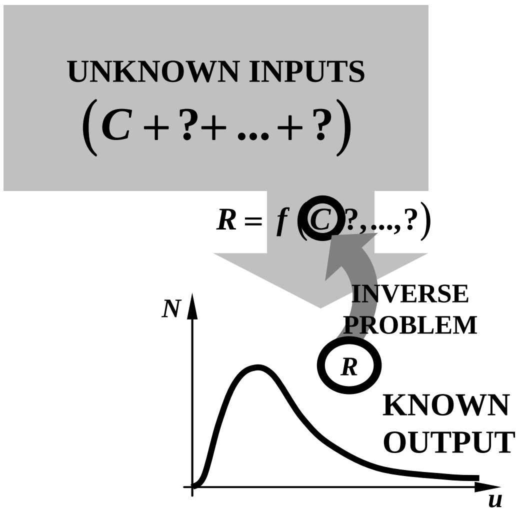

Figure 2. Schematization of an experimental test: uniaxial compression case.

To identify constitutive laws starting from experimental results, it is therefore necessary to evaluate all factors influencing a test result *R*, the known output of our identifying problem with unknown inputs (Fig. 2). Among these inputs, there is the constitutive behavior *C*, which is unknown in value. The other contributions come from the structural nature of the specimen and are unknown also in kind and number. The knowledge of *f*, the function relating *R* to all the unknown inputs (Fig. 2), is fundamental to establish a relationship between *R* and *C*, the input to identify. The *C* identification places then as a typical inverse problem. As most of the inputs are unknown in kind and number, the definition of a model is required to establish the correlation between *C* and *R*. In the model here adopted, it has been assumed that the main factors influencing *R* are four: constitutive properties (*C*), development of failure patterns changing the structural scheme (*S*), interactions between test-machine and specimen (*I*), and response time of the test-machine (*M*):

$$R = C + S + I + M \ . \qquad (2)$$

A qualitative representation of the four main factors for different load-steps is shown in Fig. 3. The repartition of *R* in Fig. 3 is consistent with the experimental evidence. Concerning this figure, it must be incidentally recalled that a stabilizing cycle is an unloading-reloading cycle effectuated for a preloading equal to about the 10% of the maximum presumed load. The stabilizing cycle is done in order to limit the influence of *I* and *M* on *R* at the beginning of the test.

It is therefore necessary to define an identifying procedure from experimental data to material behavior (inverse problem), which is not affected by the remarks concerning the approach of Fig. 1. In particular, it has been previously argued that strain-softening does not seem to be reasonable in a constitutive law. Thus, a monotonic non-decreasing law can be expected to characterize the constitutive behavior of concrete. This analysis gives us a valid reason for a critical assessment of the problem of the existence of strain-softening. As in Hudson et al. (1971), also in the present work it was supposed that strain-softening is not a material property, but is essentially due to scaling the applied force by the original cross-sectional area rather than the actual cross-sectional area.

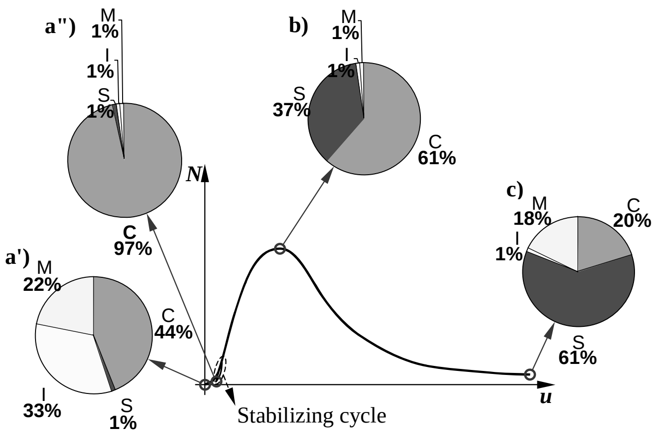

Figure 3. Qualitative repartition of *R* at the beginning without stabilizing cycle (a'), at the beginning with stabilizing cycle (a"), at an intermediate load step (b), and at the end of the test (c).

From similar considerations, Rosati and Natali Sora (2001) proposed a complete response curve for concrete loaded in tension. They also assumed that, during crack propagation, the specimen behaves similarly to a structure interacting with the test machine, and its combination of behavior is not the constitutive behavior of the material. They also stated that the descending branch measured in a direct-tension test cannot be considered as a pure material property, but is a combination of constitutive and structural behaviors. They derived the constitutive law for solid concrete by a parametric analysis. In contrast, no assumption on the formulation of the constitutive law is made in the present approach.

2.1. Resistant area and effective stress. We denote K_C, K_S, K_I, and K_M the weighed contributions to *R* assumed by *C, S, I*, and *M* (Fig. 3):

$$C = K_C R, \quad S = K_S R, \quad I = K_I R, \quad M = K_M R. \tag{3}$$

With this position, it follows that $K_C + K_S + K_I + K_M = 1$.

In the aim to evaluate the relationship between *C* and *R* (Fig. 2), we group all the weighed contributions except the constitutive one into one factor *K*:

$$K = K_S + K_I + K_M. \tag{4}$$

From the identifying procedure in Fig. 1 it follows that:

$$C \equiv R. \tag{5}$$

With the position in (4), (5) is replaced by the relationship:

$$C = K_C R = \left(1 - K_S - K_I - K_M\right) R = \left(1 - K\right) R. \tag{6}$$

The last equality in (6) allows evaluation of the constitutive properties, taking into account the behavior of the specimen-test machine system by means of *K*. This approach is formally more correct than the approach in (5). Nevertheless, it is not of immediate use for identifying constitutive properties, since $K_C = K_C\left(R\right)$, $K_S = K_S\left(R\right)$, $K_I = K_I\left(R\right)$, and $K_M = K_M\left(R\right)$ are load-step functions (Fig. 3). That is to say, *K* is a load-step function, and not a constant of the performing test:

$$K = K\left(R\right). \tag{7}$$

In conclusion, it is not possible to establish a homothetic correspondence between the experimental *N-u* relationship and the uniaxial constitutive one. Therefore, the identified curve may not have the softening behavior of the experimental curve. Moreover, *K* can only be estimated, with regard to the material scale, since it includes factors that are not directly quantifiable. The material law following from the identification process will be termed the effective, and not the constitutive, in order to emphasize the non-measurability of *K*. A qualitative analysis of K_C, K_S, K_I, and K_M in compressed concrete cylinders showed that $K \cong K_S$ except the very end of the test (Ferretti, 2001; Fig. 3). Thus, the large structural scheme variation following from the propagation of dominant bi-cone shaped cracks (Fig.

4) is preponderant in comparison to the other addends in (4). To identify the scale factor of the σ axis with respect to the N axis (Fig. 1), it is fundamental to introduce a parameter whose dimensions are those of an area and whose incremental law is linked to the structural scheme variation. In the following, this parameter will be termed the resistant area A_{res}. Thus, any specimen can be regarded as consisting of a resistant structure (Fig. 4), in which crack propagation never occurred, and a volume of incoherent material. The bi-cone shaped cracks separating the resistant structure from the incoherent material do not occur abruptly, but progressively develop from the very beginning of the test forth.

Here, we suggest to estimate the progressive percentage decrement of area due to crack propagation by means of the scalar D:

$$D = \frac{A_n - A_{res}}{A_n} . \tag{8}$$

By means of Eq. 8, the resistant area can be expressed as:

$$A_{res} = A_n \left(1 - D\right) . \tag{9}$$

The effective stress has been defined as the average stress acting on the area A_{res}:

Figure 4. Resistant structure at the end of the test.

$$\sigma_{eff} = N / A_{res} . \tag{10}$$

Alternatively, the effective stress can be expressed as:

$$\sigma_{eff} = \bar{\sigma}\, A_n / A_{res} . \tag{11}$$

3. Strain-softening existence: an analytical point of view. The sign of the effective stress derivative in the σ_{eff} - $\bar{\varepsilon}$ plane is discussed here. In (10), we make explicit the dependence of σ_{eff} , A_{res} , N e D on the displacement u:

$$\sigma_{eff}\left(u\right) = N\left(u\right) / A_{res}\left(u\right) . \tag{12}$$

Now, find the derivative of Eq. (12) with respect to the variable $\bar{\varepsilon}$:

$$d\sigma_{eff} / d\bar{\varepsilon} = \sigma'_{eff}\; du/d\bar{\varepsilon} = H\left(N' A_{res} - N A'_{res}\right) / A_{res}^2 . \tag{13}$$

The superscript indicates derivation with respect to u, and H is the gauge length:

$$u = H\bar{\varepsilon} . \tag{14}$$

Said $\hat{u}$ is the displacement corresponding to the maximum load, it follows that:

$$N\left(u\right)\Big|_{u=\hat{u}} = N_{\max} . \tag{15}$$

As to the discussion of the sign of Eq. (13), it can be stated that:

- N is monotonic non-decreasing until the peak ($N' \geq 0$, $0 \leq u \leq \hat{u}$), and monotonic strictly non-increasing beyond the peak ($N' < 0$, $u > \hat{u}$);

- A_{res} is monotonic non-increasing in all the domain ($A'_{res} \le 0$, for all u), and can have a zero tangent only in a neighborhood of the origin, corresponding to the linear elastic state of the material.

For Eq. (9), the assumption of monotonic behavior for A_{res} involves a condition of monotonic behavior for the law describing the scalar $D = D(u)$.

The experimental results agree with the condition of non zero tangent of A_{res} and D for $u = \hat{u}$, since the crack propagation rate near $u = \hat{u}$ is always very fast:

$$A'_{res}\Big|_{u=\hat{u}} \ne 0\,, \qquad D'\Big|_{u=\hat{u}} \ne 0\,. \tag{16}$$

It immediately follows that the numerator in Eq. (13) is strictly positive for $0 \le u \le \hat{u}$. Then, the sign of $d\sigma_{eff} / d\bar{\varepsilon}$ is strictly positive for $0 \le u \le \hat{u}$:

$$d\sigma_{eff} / d\bar{\varepsilon} > 0 \qquad 0 \le u \le \hat{u}\,. \tag{17}$$

In particular, for $u = \hat{u}$ the Eq. (13) assumes the value of:

$$d\sigma_{eff} / d\bar{\varepsilon}\Big|_{u=\hat{u}} = -NH\, A'_{res} / A^2_{res} > 0\,, \tag{18}$$

The strict inequality comes from (16). Eq. (17), implies the following important result: a point with strictly positive tangent in the σ_{eff} - $\bar{\varepsilon}$ curve corresponds to the point with zero tangent in the *N*-*u* curve. This is a notable result, since it is obtained without having introduced any other assumptions on the shape of the law describing the decrement of A_{res}, except the physically justifiable condition of non zero tangent in correspondence of the maximum load. It can easily be demonstrated how the same result for the sign of the tangent can be transposed to the σ_{eff} - ε_{eff} curve, at the point corresponding to the $u = \hat{u}$ point of the *N*-*u* curve.

As far as the sign of Eq. (13) for $u > \hat{u}$ is concerned, it depends on the value of ρ, the ratio between the two terms in the numerator of Eq. (13):

$$\rho = N' A_{res} / N A'_{res}\,. \tag{19}$$

The result is:

$$d\sigma_{eff} / d\bar{\varepsilon} \ge 0 \qquad \text{for all } u > \hat{u},\ 0 \le \rho \le 1 \tag{20$'$}$$

$$d\sigma_{eff} / d\bar{\varepsilon} < 0 \qquad \text{for all } u > \hat{u},\ \rho > 1\,. \tag{20$''$}$$

One can also examine the sign for $u > \hat{u}$ of the derivative of *q*, defined as follows:

$$q = \frac{\bar{\sigma}_{\max}}{\sigma_{eff}} \overset{(9)}{=} \frac{A_{res}}{A_n} \Big/ \frac{N}{N_{\max}}\,. \tag{21}$$

Thus, *q* is the ratio between the normalized resistant area and the normalized load. It follows that:

$$q' = -\bar{\sigma}_{\max}\, \sigma'_{eff} / \sigma^2_{eff} = -\bar{\sigma}_{\max} \left(N' A_{res} - N A'_{res} \right) / N^2\,. \tag{22}$$

From (22) it can be observed that the sign of q' is determined by ρ:

$$q' > 0 \qquad \text{for all } u > \hat{u},\ \rho > 1; \tag{23′}$$
$$q' \le 0 \qquad \text{for all } u > \hat{u},\ 0 \le \rho \le 1. \tag{23″}$$

On the other hand, the sign of q' follows directly from Eqs. (20) and the first equality in Eq. (22), which states that q' and σ'_{eff} have opposite signs for all u. In conclusion, the sign of $d\sigma_{eff} / d\bar{\varepsilon}$ is surely positive for $0 \le u \le \hat{u}$, whereas it is only known when the law describing D is known for $u > \hat{u}$. A proposal to experimentally acquire *D* is provided in Ferretti (2001). From these acquisitions, a monotone non-decreasing effective was found for all u.

4. Conclusions. A concrete specimen under uniaxial monotonic compression in displacement-control is characterized by load (*N*)-displacement (*u*) diagrams with softening. During loading, the specimen exhibits a crack propagation pattern depending on its structural nature and interaction with the test-machine. The *N*-*u* diagram itself is affected by the structural nature of the specimen and interaction, since crack propagation modifies the resistant structure. Considering the specimen as a structure interacting with the test-machine, it was demonstrated that the σ_{eff} - ε_{eff} curve has a strictly positive derivative at the point corresponding to the peak of the $\bar{\sigma}$ - $\bar{\varepsilon}$ curve. Contrary to what has been traditionally asserted, thus, at this point material instability does not occur. This result is independent from the law describing the decrement of resistant area. On the contrary, whether material instability actually occurs for larger strains depends on the shape of the this law.

References

Bažant, Z. P. and Belytschko, T. B. (1985). Wave Propagation in a Strain-Softening Bar: Exact Solution. *Journal of the Engineering Mechanics* **111**(3), 381–389.

Bažant, Z. P., Belytschko, T. B. and Chang, T. (1984). Continuum Theory for Strain-Softening. *Journal of the Engineering Mechanics* **110**(12), 1666–1692.

Bergan, P.G. (1983). Record of the Discussion on Numerical Modeling. *IUTAM W. Prager Symposium*, Northwestern University, Evanston, Ill.

Ferretti, E. (2001). Modellazione del Comportamento del Cilindro Fasciato in Compressione. *Ph.D. Thesis*, University of Lecce, Italy.

Ferretti, E. and Carli, R. (1999). Programma Sperimentale sul Comportamento in Compressione Monoassiale del Calcestruzzo; Parte II: Elaborazione dei Risultati Sperimentali. *Technical Note 33*, DISTART – University of Bologna, Italy.

Ferretti, E. and Di Leo, A. (2003). Modeling of Compressive Tests on FRP Wrapped Concrete Cylinders through a Novel Triaxial Concrete Constitutive Law. *Scientific Israel – Technological Advantages*, **5**, 20–43.

Hadamard, J. (1903). *Leçons sur la Propagation des Ondes* (Chapter VI), Hermann et Cie, Paris, France.

Hegemier, G.A. and Read, H.E. (1983). Some Comments on Strain-Softening. *DARPA-NSF Workshop*, Northwestern University, Evanston, Ill.

Hudson, J.A., Brown, E.T. and Fairhurst, C. (1971). Shape of the Complete Stress-Strain Curve for Rock. *13th Symposium on Rock Mechanics*, University of Illinois, Urbana, Ill.

Rosati, G. and Natali Sora, M.P. (2001). Direct Tensile Tests on Concretelike Materials: Structural and Constitutive Behaviors. *Journal of the Engineering Mechanics* **127**, 364–371.

Sandler, I. and Wright, J.P. (1983). Summary of Strain-Softening. *DARPA-NSF Workshop*, Northwestern University, Evanston, Ill.