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The second order solution of Boussinesq's problem

E. Ferretti
DICAM, School of Engineering and Architecture – University of Bologna, Bologna, Italy

ABSTRACT: In this document, we have proposed a second order solution to Boussinesq's problem (Boussinesq 1885), which allows us to account for the new experimental evidence (Ferretti & Bignozzi 2012, Ferretti 2012b) on the stress field induced by aircraft traffic in concrete pavements. In particular, the second order solution is able to describe the tensile state of stress acquired in the proximity of the contact area and not accounted for in the classical solution of Boussinesq's problem for a homogeneous linear-elastic and isotropic half-space. The second order solution also allows us to evaluate the effect of the elastic constants on the stress field, improving the solution of Boussinesq in this second case also.

1 THE FIRST ORDER ELASTIC SOLUTION

1.1 *First integral of the equilibrium problem*

The first solution of Boussinesq is based on the similarity between the equilibrium equations in terms of displacements, with the body forces per unit volume, f_x, f_y and f_z, set equal to zero:

$$\begin{cases} (\lambda+\mu)\dfrac{\partial I_{1\varepsilon}}{\partial x}+\mu\nabla^2 u=0 \\ (\lambda+\mu)\dfrac{\partial I_{1\varepsilon}}{\partial y}+\mu\nabla^2 v=0 \\ (\lambda+\mu)\dfrac{\partial I_{1\varepsilon}}{\partial z}+\mu\nabla^2 w=0 \end{cases} \quad (1)$$

where $I_{1\varepsilon}$ is the bulk strain, and the three equations:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(2\dfrac{\partial P}{\partial z}\right)-\nabla^2\dfrac{\partial(zP)}{\partial x}=0 \\ \dfrac{\partial}{\partial y}\left(2\dfrac{\partial P}{\partial z}\right)-\nabla^2\dfrac{\partial(zP)}{\partial y}=0 \\ \dfrac{\partial}{\partial z}\left(2\dfrac{\partial P}{\partial z}\right)-\nabla^2\dfrac{\partial(zP)}{\partial z}=0 \end{cases} \quad (2)$$

that are satisfied by any potential function P for which the Laplacian is equal to zero:

$$\nabla^2 P=\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}=0. \quad (3)$$

The use of the equilibrium equations with the body forces set equal to zero for studying the stress field induced in the soil by a point-load is allowed by the superposition principle: assuming a linear-elastic behavior for the soil, we can separately analyze and superpose the stress field induced in the soil by the point-load and the stress field induced in the soil by the weight of the soil itself.

The similarity between the two systems of Equations 1 and 2 is established in the assumptions:

$$u=-\frac{\partial}{\partial x}(zP),\ \ v=-\frac{\partial}{\partial y}(zP),\ \ w=-\frac{\partial}{\partial z}(zP)+KP, \quad (4)$$

$$\frac{\lambda+\mu}{\mu}I_{1\varepsilon}=2\frac{\partial P}{\partial z}. \quad (5)$$

The function P chosen by Boussinesq is the logarithmic potential Ψ for the prefixed point (x, y, z) of the semi-space under the surface, at the distance r from the point $Q \equiv (x_1, y_1, 0)$ of the load surface:

$$r=\sqrt{(x-x_1)^2+(y-y_1)^2+z^2}, \quad (6)$$

$$P=\Psi=\int \log(z+r)\,dm, \quad (7)$$

where dm is given by $\rho(x_1, y_1)$, the mass density for unit surface at the point Q:

$$\int dm=\int \rho(x_1,y_1)\,dx_1 dy_1. \quad (8)$$

The solution of the equilibrium problem in terms of potentials is:

$$(u,v)=-\frac{\partial}{\partial(x,y)}(z\Psi),\ \ w=-\frac{\partial}{\partial z}(z\Psi)+K\Psi. \quad (9)$$

Since Equations 9 satisfy the conditions of equilibrium but do not provide the right displacement field for

$z \to \infty$ (Boussinesq 1885), Boussinesq used the first derivative $\partial\Psi/\partial z$ instead of the logarithmic potential Ψ for the potential P in Equations 2, which is allowable since $\partial\Psi/\partial z$ still gives a Laplacian equal to zero:

$$P = \frac{\partial \Psi}{\partial z} = \int \frac{dm}{r}, \tag{10}$$

$$(u,v) = -z\frac{\partial}{\partial(x,y)}\frac{\partial\Psi}{\partial z}, \quad w = -z\frac{\partial^2\Psi}{\partial z^2} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial\Psi}{\partial z} \tag{11}$$

$$\begin{cases} \tau_{xz} = -2\mu\dfrac{\partial}{\partial x}\left(\dfrac{\mu}{\lambda+\mu}\displaystyle\int\dfrac{dm}{r} + z\int\dfrac{zdm}{r^3}\right) \\ \tau_{yz} = -2\mu\dfrac{\partial}{\partial y}\left(\dfrac{\mu}{\lambda+\mu}\displaystyle\int\dfrac{dm}{r} + z\int\dfrac{zdm}{r^3}\right) \\ \sigma_z = -2\mu\displaystyle\int\left(\dfrac{\mu}{\lambda+\mu}\dfrac{z}{r^3} + 3\dfrac{z^3}{r^5}\right)dm \end{cases} \tag{12}$$

For the points of the surface, Boussinesq provides the results:

$$u = 0, \quad v = 0, \quad w = \frac{\lambda+3\mu}{\lambda+\mu}\int\frac{dm}{r}, \tag{13}$$

$$I_{1\varepsilon} = -4\pi\frac{\mu}{\lambda+\mu}\rho(x,y), \tag{14}$$

$$\begin{cases} (p_x, p_y) = \lim\limits_{z\to 0} -2\dfrac{\mu^2}{\lambda+\mu}\dfrac{\partial}{\partial(x,y)}\displaystyle\int\dfrac{dm}{r} \\ p_z = 4\pi\mu\dfrac{\lambda+2\mu}{\lambda+\mu}\rho(x,y) \end{cases} \tag{15}$$

The third of Equations 15 and the first two of Equations 13 tell us that this is the case in which the boundary conditions consist in giving the normal component of the load and assuming the horizontal displacements on the surface to be equal to zero.

A discussion of the original treatment of Boussinesq (Boussinesq 1885) can be found in Ferretti (2012a), where it is pointed out how it seems unnecessary to perform integrals on the whole load surface – as Boussinesq does – since the aim of the treatment is to find the solution for a single point-load, and not for a distributed load. In effect, after obtaining the general solution, Boussinesq gives the point-load solution by substituting the integrals with their integrands, that is, by causing the dimensions of the load surface to vanish. It therefore seems possible, besides being simpler, to build the point-load solution directly, by defining the potentials for the infinitesimal superficial neighborhood of the point:

$$\psi = \log(z+r)\,dm, \tag{16}$$

$$P = \frac{\partial\psi}{\partial z} = \frac{1}{r}dm. \tag{17}$$

The solution following by the position in Equation 17 is:

$$\begin{cases} u = \dfrac{z(x-x_1)}{r^3}dm, \quad v = \dfrac{z(y-y_1)}{r^3}dm \\ w = \dfrac{r^2(\lambda+3\mu)+z^2(\lambda+\mu)}{r^3(\lambda+\mu)}dm \end{cases} \tag{18}$$

$$I_{1\varepsilon} = -\frac{2\mu}{\lambda+\mu}\frac{z}{r^3}dm, \tag{19}$$

$$\begin{cases} \tau_{xz} = 2\mu(x-x_1)\dfrac{\mu r^2 + 3(\lambda+\mu)z^2}{(\lambda+\mu)r^5}dm \\ \tau_{yz} = 2\mu(y-y_1)\dfrac{\mu r^2 + 3(\lambda+\mu)z^2}{(\lambda+\mu)r^5}dm \\ \sigma_z = -2\mu z\dfrac{\mu r^2 + 3(\lambda+\mu)z^2}{(\lambda+\mu)r^5}dm \end{cases} \tag{20}$$

For $z \to 0$ (points of the surface), we find:

$$u = 0, \quad v = 0, \quad w = \frac{\lambda+3\mu}{r(\lambda+\mu)}dm \tag{21}$$

$$I_{1\varepsilon} = \begin{cases} 0 & \text{for } r > 0 \\ -\dfrac{2\mu}{\lambda+\mu}\rho(x_1,y_1) & \text{for } r \to 0 \end{cases}, \tag{22}$$

$$\begin{cases} p_x = \dfrac{2\mu^2}{\lambda+\mu}\dfrac{x-x_1}{r^3}dm, \quad p_y = \dfrac{2\mu^2}{\lambda+\mu}\dfrac{y-y_1}{r^3}dm \\ p_z = \begin{cases} 0 & \text{for } r > 0 \\ 2\mu\dfrac{\lambda+2\mu}{\lambda+\mu}\rho(x_1,y_1) & \text{for } r \to 0 \end{cases} \end{cases} \tag{23}$$

1.2 *Second integral of the equilibrium problem*

The second solution follows from the position:

$$u = \frac{\partial P}{\partial x}, \quad v = \frac{\partial P}{\partial y}, \quad w = \frac{\partial P}{\partial z}. \tag{24}$$

Due to Equation 3, in this second case the bulk strain is equal to zero:

$$I_{1\varepsilon} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla^2 P = 0, \tag{25}$$

and the equilibrium equations expressed by Equations 1 are identically satisfied.

By using the logarithmic potential given in Equation 16 instead of Ψ, the second solution of Boussinesq is substituted by the following (Ferretti 2012a):

$$\begin{cases} u = \frac{\partial \psi}{\partial x} = \frac{x - x_1}{r(z+r)} dm \\ v = \frac{\partial \psi}{\partial y} = \frac{y - y_1}{r(z+r)} dm \\ w = \frac{\partial \psi}{\partial z} = \frac{1}{r} dm \end{cases} \tag{26}$$

$$\begin{cases} (\tau_{xz}, \tau_{yz}) = -2\mu \frac{\partial}{\partial(x,y)} \left(\frac{1}{r}\right) dm \\ \sigma_z = 2\mu \frac{\partial}{\partial z}\left(\frac{1}{r}\right) dm = -2\mu \frac{z}{r^3} dm \end{cases} \tag{27}$$

and, for the points of the surface:

$$\begin{cases} u = \frac{\partial \psi}{\partial x} = \frac{x - x_1}{r^2} dm \\ v = \frac{\partial \psi}{\partial y} = \frac{y - y_1}{r^2} dm \\ w = \frac{\partial \psi}{\partial z} = \frac{1}{r} dm \end{cases} \tag{28}$$

$$\begin{cases} p_x = 2\mu \frac{x - x_1}{r^3} dm, \quad p_y = 2\mu \frac{y - y_1}{r^3} dm \\ p_z = \langle \begin{matrix} 0 & \text{for } r > 0 \\ 2\mu\rho(x_1, y_1) & \text{for } r \to 0 \end{matrix} \end{cases} \tag{29}$$

1.3 *Third integral of the equilibrium problem*

For the third solution, Boussinesq assumes:

$$u = -\frac{\partial P}{\partial y}, \quad v = \frac{\partial P}{\partial x}, \quad w = 0 \tag{30}$$

which are the plane strain conditions.

As for the second solution, in this case also, the bulk strain is equal to zero and the equilibrium equations are identically satisfied, due to Equation 3

Since the bulk strain is equal to zero, the condition of plane strain implies plane stress in each point of the body. The stress field is given by:

$$\tau_{xz} = \mu \frac{\partial^2 P}{\partial y \partial z}, \quad \tau_{yz} = -\mu \frac{\partial^2 P}{\partial x \partial z}, \quad \sigma_z = 0, \tag{31}$$

where:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0. \tag{32}$$

This is the solution for the case in which the normal component of the external load is set equal to zero and the two shear components stand in the relationship represented by Equation 32. The displacements take place horizontally.

1.4 *Elastic solution for a point-load perpendicular to the surface*

Due to the superposition principle, it is always possible to find further solutions to the equilibrium problem by combining the former three solutions with each other. Along these lines, Boussinesq formed two linear combinations of Equations 11 and 24. The linear combination giving the solution of the point-load perpendicular to the surface is obtained by multiplying Equations 11 by the inverse of $-4\pi\mu$ and Equations 24 by the inverse of $-4\pi(\lambda + \mu)$. The multiplying factors of the linear combination have been chosen specifically to equal p_z and $\rho(x, y)$:

$$p_z = \lim_{z \to 0} -\frac{1}{2\pi} \frac{\partial^2 \Psi}{\partial z^2} = \rho(x, y). \tag{33}$$

and, consequently, also the external load dF anddm:

$$dF = p_z dx_1 dy_1 = \rho(x_1, y_1) dx_1 dy_1 = dm. \tag{34}$$

Due to the superposition principle, the same multiplying factors can be taken to build a second linear combination, providing the solution in terms of stresses. In the case of infinitesimal load surface, the well-known solutions of Boussinesq for a point-load perpendicular to the surface is:

$$\begin{cases} \tau_{xz} = \frac{3}{2\pi} \frac{z^2}{r^5} (x - x_1) dF \\ \tau_{yz} = \frac{3}{2\pi} \frac{z^2}{r^5} (y - y_1) dF \\ \sigma_z = -\frac{3}{2\pi} \frac{z^3}{r^5} dF \end{cases} \tag{35}$$

where the stresses are independent of the elastic coefficients of the medium.

In Ferretti (2012a), the elastic solution for a point-load perpendicular to the surface has been derived as linear combination of Equations 18 and 26, for displacements, and of Equations 20 and 27, for stresses, providing:

$$\begin{cases} u = \frac{x - x_1}{2\mu} \left[\frac{z}{r^3} - \frac{\mu}{\lambda + \mu} \frac{1}{r(z+r)} \right] dF \\ v = \frac{y - y_1}{2\mu} \left[\frac{z}{r^3} - \frac{\mu}{\lambda + \mu} \frac{1}{r(z+r)} \right] dF \\ w = \frac{1}{2\mu r} \left[\frac{\lambda + 2\mu}{\lambda + \mu} + \frac{z^2}{r^2} \right] dF \end{cases} \tag{36}$$

$$I_{1\varepsilon} = \frac{1}{\lambda+\mu}\frac{d}{dz}\left(\frac{1}{r}\right)dm = -\frac{1}{\lambda+\mu}\frac{z}{r^3}dF\,, \quad (37)$$

$$\begin{cases} \tau_{xz} = 3\dfrac{z^2}{r^5}(x-x_1)dF \\ \tau_{yz} = 3\dfrac{z^2}{r^5}(y-y_1)dF \\ \sigma_z = -3\dfrac{z^3}{r^5}dF \end{cases} \quad (38)$$

for a point inside the soil, and

$$\begin{cases} u = -\dfrac{1}{2(\lambda+\mu)}\dfrac{x-x_1}{r^2}dP \\ v = -\dfrac{1}{2(\lambda+\mu)}\dfrac{y-y_1}{r^2}dP \\ w = \dfrac{\lambda+2\mu}{2\mu(\lambda+\mu)}\dfrac{1}{r}dP \end{cases} \quad (39)$$

$$I_{1\varepsilon} = \begin{cases} 0 & \text{for } r>0 \\ -\dfrac{1}{\lambda+\mu}\rho(x_1,y_1) & \text{for } r\to 0 \end{cases} \quad (40)$$

$$\begin{cases} p_x = 0, \quad p_y = 0 \\ p_z = \begin{cases} 0 & \text{for } r>0 \\ \lim\limits_{z\to 0} -\dfrac{\partial^2\psi}{\partial z^2} = \rho(x_1,y_1) & \text{for } r\to 0 \end{cases} \end{cases} \quad (41)$$

for the points of the surface.

2 THE SECOND ORDER ELASTIC SOLUTION

2.1 *A second order solution of the first integral*

Following the spirit of the superposition principle and noting that the partial derivatives of any arbitrary order of the function ψ, defined in Equation 16, have a zero Laplacian (i.e. they satisfy the condition $\nabla^2 = 0$), it is possible to refine the elastic solution of Boussinesq by adding to it a further solution of Equations 2, obtained by substituting ψ with one of its derivatives of second order. This observation will be used here in order to find a further form of the first integral, which, combined to the former form and the second integral, could provide a stress solution to the vertical point-load problem that also depends on the elastic constants of the soil.

Assuming:

$$P = \frac{\partial^2\psi}{\partial z^2} = -\frac{z}{r^3}dm\,, \quad (42)$$

we find:

$$\begin{cases} u = -3\dfrac{z^2(x-x_1)}{r^5}dm \\ v = -3\dfrac{z^2(y-y_1)}{r^5}dm \\ w = -\dfrac{z}{r^3}\left(3\dfrac{z^2}{r^2} + \dfrac{2\mu}{\lambda+\mu}\right)dm \end{cases} \quad (43)$$

$$I_{1\varepsilon} = \frac{2\mu}{\lambda+\mu}\left(3\frac{z^2}{r^5} - \frac{1}{r^3}\right)dm\,, \quad (44)$$

$$\begin{cases} \tau_{xz} = \dfrac{6\mu}{\lambda+\mu}\dfrac{(x-x_1)z}{r^5}\left[\lambda - 5(\lambda+\mu)\dfrac{z^2}{r^2}\right]dm \\ \tau_{yz} = \dfrac{6\mu}{\lambda+\mu}\dfrac{(y-y_1)z}{r^5}\left[\lambda - 5(\lambda+\mu)\dfrac{z^2}{r^2}\right]dm \\ \sigma_z = -2\mu\left(\dfrac{\lambda+2\mu}{\lambda+\mu}\dfrac{1}{r^3} + \dfrac{6\lambda+3\mu}{\lambda+\mu}\dfrac{z^2}{r^5} - 15\dfrac{z^4}{r^7}\right)dm \end{cases} \quad (45)$$

inside the soil, and:

$$u = 0, \quad v = 0, \quad w = 0, \quad (46)$$

$$I_{1\varepsilon} = \begin{cases} -\dfrac{2\mu}{\lambda+\mu}\dfrac{1}{r^3}dm & \text{for } r>0 \\ -\infty & \text{for } r\to 0 \end{cases} \quad (47)$$

$$\begin{cases} p_x = 0, \quad p_y = 0 \\ p_z = \begin{cases} 2\mu\dfrac{\lambda+2\mu}{\lambda+\mu}\dfrac{1}{r^3}dm & \text{for } r>0 \\ \infty & \text{for } r\to 0 \end{cases} \end{cases} \quad (48)$$

for $z \to 0$.

Since p_x and p_y are equal to zero, Equations 45 may be added to the combined solution of first order without changing the nature of the solved problem, which still is a vertical point-load problem. Moreover, due to the infinite value achieved by p_z for $z, r \to 0$, the second form of the first integral seems to be useful for building the combined solution in all the points of the soil apart from the one of load application.

2.2 *Combined solution*

The combined solution of second order is built by multiplying Equations 45 for $-C$ and adding the result to Equations 38. The resulting stress field (for $r \neq 0$) is now in relationship with the elastic constants of the soil:

$$\begin{cases} \tau_{xz} = 3\dfrac{(x-x_1)z}{r^5}\left[z+2C\mu\left(5\dfrac{z^2}{r^2}-\dfrac{\lambda}{\lambda+\mu}\right)\right]dm \\ \tau_{yz} = 3\dfrac{(y-y_1)z}{r^5}\left[z+2C\mu\left(5\dfrac{z^2}{r^2}-\dfrac{\lambda}{\lambda+\mu}\right)\right]dm \\ \sigma_z = -\dfrac{1}{r^3}\left[3\dfrac{z^3}{r^2}+2C\mu\left(15\dfrac{z^4}{r^4}-3\dfrac{2\lambda+\mu}{\lambda+\mu}\dfrac{z^2}{r^2}-\right.\right. \\ \left.\left.\qquad -\dfrac{\lambda+2\mu}{\lambda+\mu}\right)\right]dm \end{cases} \tag{49}$$

As far as the third of Equations 49 is concerned, we may easily verify that the new terms (the ones which are multiplied for C) significantly modify the normal stress when approaching the surface, while they are negligible at great depths. Indeed, for $z \to 0$:

$$-p_z = \lim_{z\to 0}\sigma_z = 2C\mu\frac{\lambda+2\mu}{\lambda+\mu}\frac{1}{r^3}dm, \tag{50}$$

while, for $z \to \infty$, the third of Equations 49 gives the combined solution of Section 1.4.

$$\lim_{z\to\infty}\sigma_z = -3\frac{z^3}{r^5}dm. \tag{51}$$

From the comparison between Equations 50 and 51, it is clear that, for $C > 0$, the normal stress for $z \to 0$ is opposite in sign to the normal stress for $z \to \infty$:

$$\text{sign}\left(\lim_{z\to 0}\sigma_z\right) = -\text{sign}\left(\lim_{z\to\infty}\sigma_z\right). \tag{52}$$

Therefore, near to the surface, the compressed soil is subjected to a normal stress of traction. This is a result not accounted for in the solution of Boussinesq and, together with the dependence of σ_z on the elastic constants, represents the most important novelty of the combined solution of second order.

From Equation 52 we can also argue that, as σ_z is a continuous function, there exists a finite value of depth for which the normal stress is equal to zero. Setting $z_0 = z_0\,(r, C, \lambda, \mu)$, the function giving the depth for which $\sigma_z = 0$, from Equations 49 we find the relationship:

$$z_0^2\left(10C\mu\frac{z_0^2}{r^4}+\frac{z_0}{r^2}-2C\mu\frac{2\lambda+\mu}{\lambda+\mu}\frac{1}{r^2}\right)= \\ =\frac{2}{3}C\mu\frac{\lambda+2\mu}{\lambda+\mu} \tag{53}$$

in which the banal solution:

$$\sigma_z = 0 \quad \text{for } r \to \infty, \tag{54}$$

has been eliminated.

As can be easily verified, for $z \to \infty$ the combined solution of second order is equal to the solution of Boussinesq even for the displacement field.

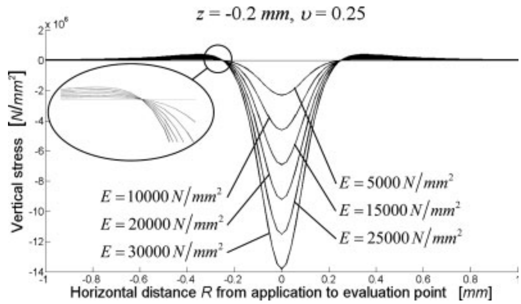

Figure 1. Parametric analysis on the Young modulus, E.

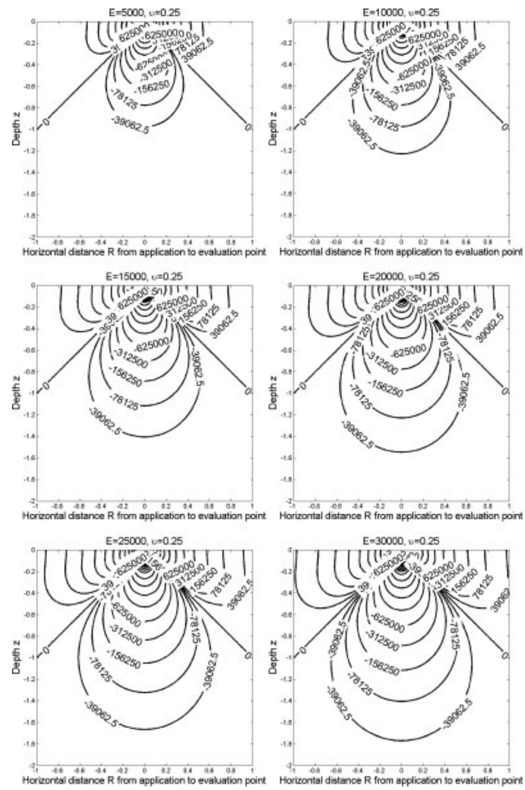

Figure 2. Parametric analysis on the Young modulus, for the vertical stress contours on the vertical cross-section (all distances are in *mm*).

3 NUMERICAL RESULTS

3.1 *Point-load perpendicular to the surface*

The plots of the vertical stress and the vertical stress contours of second order for a prefixed Poisson's ratio, υ, and variable values of Young's modulus, E, are given in Figure 1 for a plane near to the surface and in Figure 2 for a vertical cross-section passing through the point load, respectively. The parametric analysis on the Poisson's ratio for a prefixed E is shown in Ferretti (2012a).

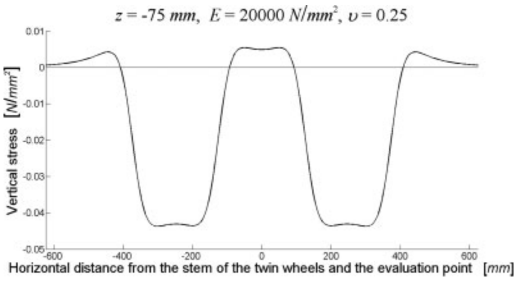

Figure 3. Vertical stress under the twin wheels of an aircraft for rectangular contact areas and uniform load.

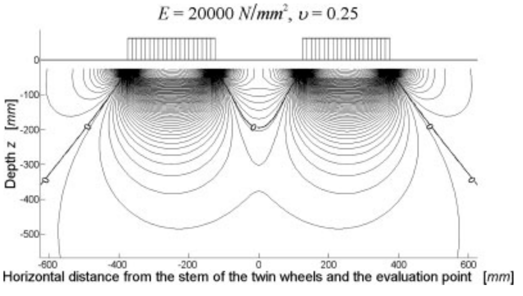

Figure 4. Vertical stress contours on the vertical cross-section for rectangular contact areas and uniform load (twin wheels).

The numerical solution of second order shows two positive peaks of vertical stress in the proximity of the application point of compression load (Fig. 1), in total agreement with the experimental data for vehicular loading shown in Ferretti & Bignozzi (2012) and Ferretti (2012b). This result gives a numerical proof that a tensile state of stress arises on the surface of soils and pavements when subjected to compression loads, with the point in which the vertical stress change in sign that is also a point in which the vertical stress does not depend upon the value of E (Fig. 1). Since both soil and concrete are assumed as not being resistant to traction – to be on the safe side – the tensile state of stress must be considered with particular attention in these materials.

Due to the tensile state of stress, there exist two families of stress contours (Fig. 2): one family for the tensioned soil and one family for the compressed soil, with the two families separated by straight lines. Moreover, the parametric analysis on the vertical cross-section shows that greater values of E increase the vertical stresses at each depth without modifying the shape of the iso-lines of stress, which change in size homothetically (Fig. 2).

3.2 *Distributed load perpendicular to the surface*

In order to evaluate which the interaction effect is of two adjacent loaded areas, the vertical stress and the vertical stress contours of second order for a vertical load that is uniformly distributed over two contact areas of rectangular shape have been plotted in Figures 3, 4, respectively. All the dimensions in Figures 3, 4 are those typical of two twin wheels in an aircraft. The existence of a tensile state of stress near the wheels is well evident even in this last case. Moreover, the interaction effect make particularly severe the positive stresses between the wheels.

Further results for contact areas of circular, rectangular and elliptic shape together with uniform and parabolic laws of external pressure distribution may be found in Ferretti (in prep.).

4 CONCLUSIONS

In this paper, we have discussed Boussinesq's solution in the light of new experimental findings on the stress distribution in a half-space subjected to point-loads. The original work of Boussinesq has been extended to provide a second order solution.

The second order solution has shown that a compression load always generates positive stresses at the surface, in the proximity of the load. The existence of a tensile state of stress, not accounted for in Boussinesq's solution, could explain the several observed mechanisms of premature damage that affect concrete pavements subjected to vehicular loading, particularly when twin wheels are involved.

The second order solution also allows us to evaluate the effect of the elastic constants on the stress field, which, in this second case also, is an improvement to Boussinesq's solution.

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