A Discrete Model to Estimate the Effective Properties of a 2D Solid with Random Voids

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Abstract

A discrete model for the estimation of the effect of random voids on the structural properties of a two-dimensional solid is presented. Triangular void cells are used to simulate randomly located micro-cracks.

The proposed model is solved using Cell Method, a recent numerical method that allows a direct discrete formulation of balance equations. Both heterogeneities of the structure and stress concentrations due to voids distribution are taken into account by the model. Following an introduction, some aspects of Cell Method for plane elasticity that are relevant for this paper will be briefly recalled and results from simulations in the elastic field will be discussed. Next, the proposed model will be extended to plastic field and more simulations will be presented. Results show that this model can be successfully employed to assess the structural response of a 2D solid with randomly distributed voids.

1. Introduction.

It is well known that the presence of distributed micro-cracks affects the mechanical behaviour of materials. In the attempt to characterize material properties when a low concentration of voids is present, a number of different approaches have been proposed; leading to analytical, beam lattice, statistical, FEM models.

For example, a simple expression for the elastic properties of a material with identical polygonal randomly oriented voids and the numerical values of the parameters to be employed for different void shapes have been determined in [1] and [2]. A beam lattice model was employed in [3] to assess the effective elastic properties of an elastic sheet with circular voids, determining that the effective elastic modulus in the direction of the traction does not depend on Poisson's modulus. Moreover, a little effort has been done to estimate the stiffness of a 2D initially elastic continuum containing square perforations using finite element models [4]. However, quoting [4, pg. 361], "The prediction of the macro-response of a diluted system in the region within which neither the mean field nor the percolation models are applicable is a difficult task which has only recently attracted the attention it deserves...Large scale mesh-insensitive simulations are still not available. The considerations of defects with different orientations, size and shapes has yet to be attempted.".

The purpose of this paper is to introduce a new numerical model for the investigation of the effect of randomly located voids on the structural response of a material. The proposed model consists of an homogeneous matrix of cells, which contains randomly distributed cracks, namely void cells, as shown in Figure 1. This model can be solved with Cell Method (CM in the following).

2. Cell Method.

CM is a recently introduced numerical method [5] that may be regarded as an alternative to Finite Element Method. It is currently being applied to several fields, including fracture mechanics, acoustics, electromagnetism, mechanics of porous materials, and so on [6-12].

The bases of CM come from a classification of variables. In any field problem, variables can be of three kinds: *configuration variables* (such as nodal displacements, strain tensor and so on), *source variables* (such as forces, torques, stress tensor, and so on) and *energy variables* (that will not be used in the following). Consistently with this classification, two staggered complexes of cells will constitute the geometrical entities to which different variables are to be associated: configuration variables will be linked to a primal complex of cells, while a dual complex will be used for source variables.

In this work, the primal cells are three nodes triangles which come from a Delaunay tessellation and the dual cells are the Voronoi mesh associated with the primal complex, as shown in Figure 2. This choice corresponds to the assumption of linear interpolation functions for the displacement field, although it should be pointed out that also higher order approximation are possible with CM. It should be also noted that there is always one node inside each dual cell.

With a linear interpolation of the displacement field, strain components will be uniform within each primal cell and expressed as

$$\{\varepsilon\}_c = [B]_c \{u\}_c \tag{1}$$

where

$\{\varepsilon\}_c = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T$,

$$[B]_c = -\frac{1}{2A_c t}\begin{bmatrix} A_{1x} & 0 & A_{2x} & 0 & A_{3x} & 0 \\ 0 & A_{1y} & 0 & A_{2y} & 0 & A_{3y} \\ A_{1y} & A_{1x} & A_{2y} & A_{2x} & A_{3y} & A_{3x} \end{bmatrix},$$

$\{u\}_c$ collects the nodal displacements,

t is the thickness of the sample,

A_c is the area of the cell,

A_{ij} is given by the projection of area A_i along j axis, as shown in Figure 3.

Stress components $\{\sigma\}_c = \{\sigma_x \quad \sigma_y \quad \tau_{xy}\}^T$ are also uniform within each primal cell and, introducing the appropriate constitutive matrix $[D]_c$, which expresses Hooke's law for the constituent cells, are given by

$$\{\sigma\}_c = [D]_c\{\varepsilon\}_c = [D]_c[B]_c\{u\}_c. \qquad (2)$$

Equations (1) and (2) are similar to those employed by FEM, but the two methods deeply differ in how balance equations are obtained. In fact, one important aspect of Cell Method is that it allows a *direct discrete formulation of physical laws*: Cell Method recognizes that the dual cell constitutes an influence region of each node and writes equilibrium equations for each dual cell.

In fact, it is possible to express the forces acting on each side of the dual cell as a function of stress components $\{\sigma\}_c$ (see Figure 4):

$$\{T\}_c = -tA_c[B]_c^T\{\sigma\}_c = tA_c[B]_c^T[D]_c[B]_c\{u\}_c \qquad (3)$$

Adding up the contributions from all the primal cells surrounding the node and the external forces acting on the dual cell itself, a balance is written for each dual cell (see Figure 5):

$$\{T_h\} + \{F_h\} = 0 \qquad (4)$$

A linear system of equations is therefore obtained, which can be written in the familiar form

$$[K]\{u\}=\{F\} \tag{5}$$

and solved with the usual methods.

This fundamental system has been directly obtained in a discrete form, without resorting to a differential formulation to form the field equation, as balance is not written for a point – the node – like in FEM, but for a whole influence region of the node, namely the dual cell.

Thus, the method is applicable when discontinuities are present, and the constitutive matrix may vary freely from one cell to another; while generally the use of FEM implies the size of the mesh to be smaller than any characteristic length involved in the problem [13].

It must be also mentioned that some aspects of the Cell Method may recall the Finite Volumes Method or the so-called *direct* – or *physical* – approach [14]. These methods did not have much success, maybe due to an intrinsic difficulty to develop higher order elements.

On the contrary, higher order interpolation functions are easily implemented in CM. A more detailed description of the method for plane elasticity and comparison with results obtained with FEM may be found in [15]. As a general statement, it can be said that CM results agree with those obtainable with other numerical methods, and that convergence and accuracy with CM are equal and sometimes even better than those achievable with FEM.

The work here presented makes use only of linear interpolation functions and focuses on the other mentioned peculiar aspect of the Cell Method: the possibility to have heterogeneities the same size of that of the mesh.

3. Simulations in elastic field.

In a first set of simulations, a model with a total number of cells N = 1498 was employed. In the model, a number n of randomly distributed cells was set to be void (see Figure 1) to account for damage. A tensile test was simulated on the model, imposing a displacement on the right end of the sample while constraining the left end, and allowing lateral contractions.

Different void densities were considered, and 5 simulation were run for each damage density. Elastic parameters for the unimpaired, ferrous, material were assumed to be E = 210 GPa (Young modulus) and ν = 0.3 (Poisson's ratio). Effective Young modulus E^* of the damaged models was computed, and results are shown in Figure 6, where $f=n/N$ (n = number of void cells, N = total number of cells in the model). Fluctuations of computed effective elastic modulus can be observed for each value of porosity, due to a structural effect derived from the different location of the randomly distributed void cells in the simulations. The linear fit, also plotted in the graph, has slope –2.47, which is similar to, but less pronounced than, the one reported for FEM simulations with square perforations in [4, pp. 360-361].

Stress concentrations near the void cells in one of the simulations are shown in Figure 7.

In an other set of simulations, results in assessment of elastic modulus with a similar model at relatively high porosities are in good agreement with experimental data for sintered alloys, as discussed in [11].

Considering in greater detail the condition of low void concentration, Figure 8 shows the ratio *E*/E* for 25 simulations on a 1498 matrix. As expected, smaller fluctuations of computed effective elastic modulus can be observed, as the influence of voids location becomes less important for low void concentrations. The linear fit for simulations results has slope 2.415.

The graph also shows the quicker decrease of stiffness which characterizes the trend assumed by the self-consistent model proposed in [3], with slope 4.2.

4. Plasticity.

Plasticity may be implemented in the framework of Cell Method. An elastic-perfectly plastic material has been assumed, which for an uni-dimensional stress state may be modeled as (see also Figure 9)

$$\varepsilon = \frac{\sigma}{E}, \qquad \lambda = 0 \quad \text{when} \quad 0 \le \sigma < \sigma_y$$

$$\sigma = E(\varepsilon - \lambda) \quad \text{when} \quad \sigma = \sigma_y \qquad (6)$$

where σ_y is the material yeld strenght.

Similar equations hold for a multi-dimensional stress state, and the uniform stress state inside a primal cell may be written as

$$\{\sigma\}_c = [D]_c(\{\varepsilon\}_c - \{\lambda\}_c) \qquad (7)$$

where $\{\lambda\}_c$ is the homogeneous inelastic strain tensor for the primal cell *c*.

Following a classical approach to the solution of non-linear elastic problems, load history has been divided in a convenient number of finite steps. The material behaviour is non linear elastic within each step. Displacements and internal stresses are known at

the beginning of each step, and a backward difference integration scheme has been employed.

Eq. (3) in incremental terms now reads

$$\{\Delta T\}_c = -[K]_c\{\Delta u\}_c + [L]_c\{\Delta\lambda\}_c \qquad (8)$$

where

- for plane stress:

$$[L]_c = tA_c[B]_c^T \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu^2}{2(1+\nu)} \end{bmatrix},$$

- for plane strain

$$[L]_c = tA_c[B]_c^T \frac{E}{(1-\nu)(1-2\nu)}\begin{bmatrix} 1-2\nu & 0 & 0 \\ 0 & 1-2\nu & 0 \\ 0 & 0 & \dfrac{1}{2}-\nu \end{bmatrix},$$

and the fundamental equation in incremental terms now becomes

$$[K]\{\Delta u\} = \{F\} + [L]\{\Delta\lambda\}. \qquad (9)$$

At step one:

$$\{\Delta\lambda\} = 0 \qquad (10)$$

and the fundamental system can be solved for $\{\Delta u\}$. The stress tensor $\{\sigma\}_c$ and the deviatoric strain tensor $\{s\}_c$ for this solution can be then computed.

Von Mises criterion has been assumed and the radius of the yeld locus considered. For each primal cell, if

$$r_{pred} = \sqrt{s_x^2 + s_y^2 + s_z^2 + s_{xy}^2} \le r_0 = \sqrt{\frac{2}{3}}\sigma_Y \qquad (11)$$

the cell is in elastic field, and $\{\Delta u\}$ is the solution. On the contrary, if

$$r_{pred} = \sqrt{s_x^2 + s_y^2 + s_z^2 + s_{xy}^2} > r_0 = \sqrt{\frac{2}{3}}\sigma_Y \qquad (12)$$

the cell is in plastic field and $\Delta r = r_{pred} - r_0$ is used to update $\{\Delta\lambda\}_c$ and $\{s\}_c$.

The process is repeated until convergence is obtained for all the cells, then a new step is considered.

As a benchmark, the problem of a beam subject to bending, for which a closed form solution can be found in [16], was considered. As discussed in [17], convergence rate of the solution in elastic field was found to be 2 – that is the same value as expected for triangular three nodes elements in FEM –, while in plastic field the rate of convergence of the computed towards the exact solution was 1.6.

Results of a simulation running a model consisting of 1498 primal cells with 38 randomly distributed void cells are displayed below. For the unimpaired material, E=210 GPa and yield strength Ry = 430 MPa were assumed.

Figure 10 shows the progressive plasticization of the model during the simulation. The simulation result is shown in Figure 11 in the form of a stress-strain plot, where the simulated material exhibited an effective modulus E^* = 201.3 GPa in the initial, macroscopically elastic, part of the diagram.

Figure 11 also displays the stress-strain curve obtained from a simulation running an unimpaired model (without voids), in which for each cell E = 210 GPa.

5. Conclusions.

In this paper a new discrete model has been employed to estimate the effect of randomly located micro-cracks on the structural response of a material.

The model consists of a homogeneous matrix in which randomly distributed cracks are present, considers both heterogeneities of the structure and stress concentrations, and is solved with Cell Method.

A brief description of Cell Method for plane problems is also given in the paper, where both elastic and elastic-plastic behaviours have been included. Results of simulations have been discussed for an elastic-perfectly plastic material with different damage levels.

Other rehological behaviours are currently being developed. For example, an incremental model including linear hardening has been recently developed and applied to porous – sintered – ferrous alloys modeling, showing in a first set of simulations a good agreement with experimental data [18].

REFERENCES

1. Jasiuk, I., Chen, J. and Thorpe, M.F., Appl. Mech. Rev., Vol.47, S18-S28, (1994)
2. Kachanov, M., Tsukrov, I. and Shafiro, B., Effective Moduli of Solids with cavities of Various Shapes, in: Micromechanics of Random Media, M. Ostoja-Starzewski and I. Jasiuk, Eds. ASME Book No AMR139, S151-S174, (1994)
3. Day, A.R., Snyder, K.A., Garboczi, E.J. and Thorpe, M.F., J. Mech. Phys. Solids, Vol. 40, 1031-1051, (1992)
4. Krajcinovic, D., Damage Mechanics, Elsevier, North Holland, (1996)
5. Tonti, E., A Direct Discrete Formulation of Field Laws: The Cell Method, CMES, vol.2, n.2, 237-258, (2001)
6. Ferretti E.,Viola E. and Di Leo A, Modellazione del comportamento macroscopico in compressione del calcestruzzo a partire da nuove assunzioni costitutive, in Proc. of XXIIX AIAS, Vicenza, Italy, (1999)
7. Nappi A., Rajgelj S. and Zaccaria, D., A discrete formulation applied to crack growth problems, in Proc. Meso Mechanics 2000 Conference, Xi'an, China, (2000)
8. Tonti E., A Finite Formulation for the Wave Equation, J. Comp. Acoustics, vol.9, n. 4, 1355-1382 (2001)
9. Tonti E., Finite Formulation of Electromagnetic Field, J. Electromagnetic Waves and Applications, PIER 32, Special Volume on Geometrical Methods for Comp. Electromagnetics, 1-44, (2001)

10. Marrone M., Computational Aspects of Cell Method in Electrodynamics, J. Electromagnetic Waves and Applications, PIER 32, Special Volume on Geometrical Methods for Comp. Electromagnetics, 317-356, (2001)
11. Cosmi F., Di Marino F., Modelling of the Mechanical Behavior of Porous Materials: A New Approach, Acta of Bioengineering and Biomechanics, 3, (2001)
12. Cosmi F., Dreossi D., Vittur F., Bone Mechanical Properties Estimation with the Cell Method, Proc. 18th DAS on Experimental Methods in Solid Mechanics, Steyr, Austria, (2001)
13. Roux, S., Continuum and Discrete Description of Elasticity, in Statistical Models for the Fracture of Disordered Media, Herrmann H.J., and Roux, S. (Eds). Elsevier, North Holland, 109-113, (1990)
14. Huebner, K.H., The finite element method for Engineers, John Wiley & Sons, 39-59, (1995)
15. Cosmi, F., CMES, vol.2, n.3, 365-372 ,(2001)
16. Hill, R., The Mathematical Theory of Plasticity, Oxford University Press, Oxford, GB, 79-81, (1956)
17. Cosmi, F., Elasto-plasticità con il Metodo delle Celle, in Proc. of XXX AIAS, Alghero, Italy, (2001)
18. Cosmi, F., Cell Method Simulation of Sintered Alloys Mechanical Behaviour, in Proc. of TC15-Imeko, Bertinoro (Forlì), Italy, 33-34, (2002)

FIGURES

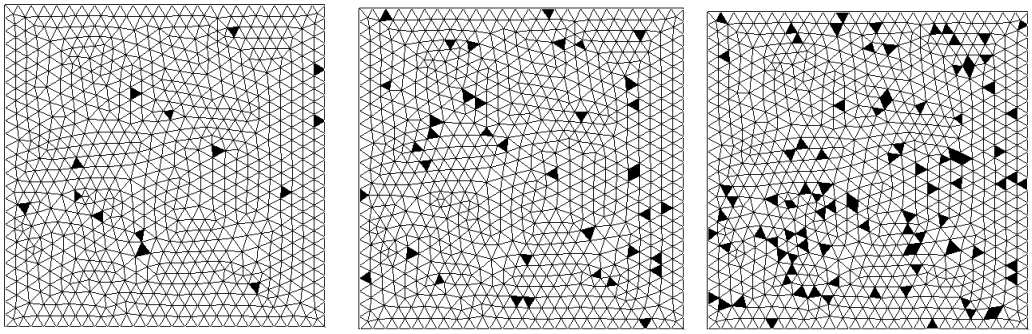

Figure 1: Randomly distributed void cells in homogeneous matrixes.

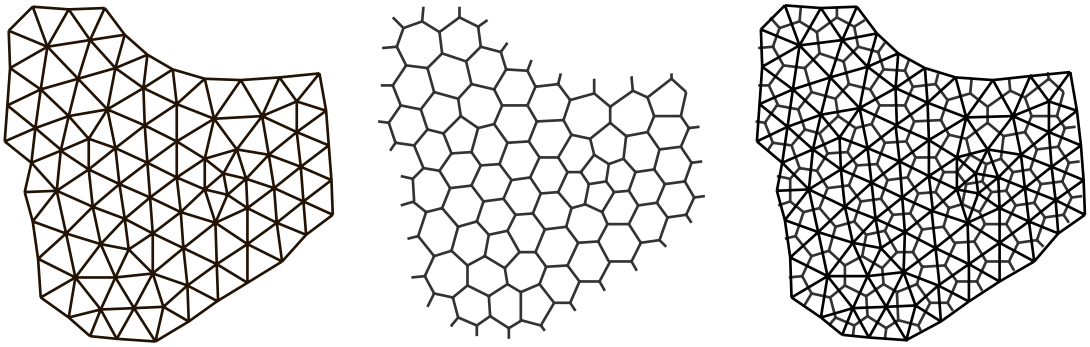

Figure 2: Primal (Delaunay), dual (Voronoi) complexes of cells and their ensemble.

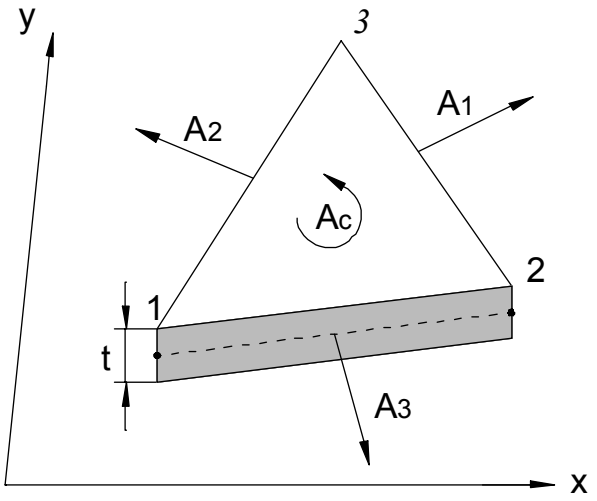

Figure 3: Geometrical quantities.

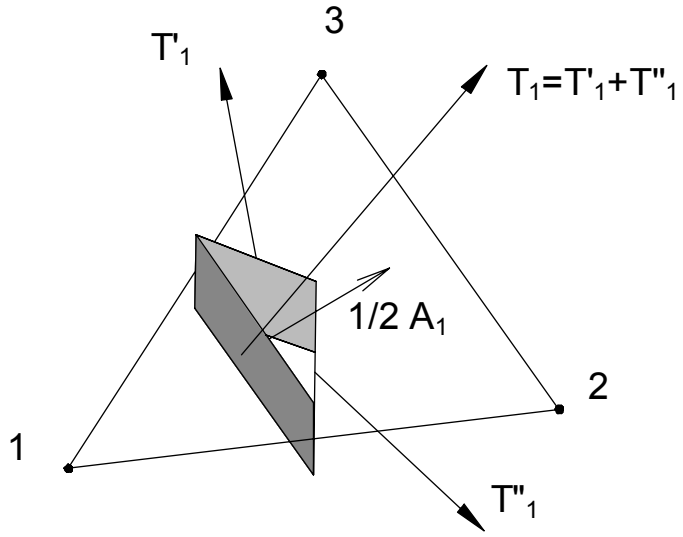

Figure 4: Forces on one side of dual cell.

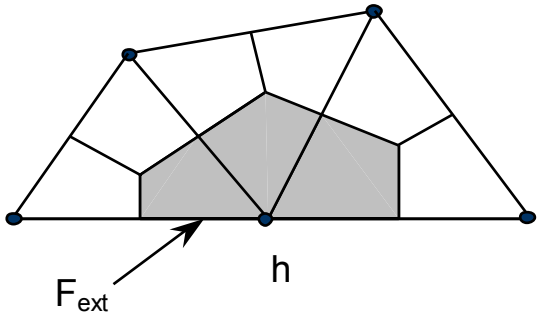

Figure 5: Balance of node *h* influence region.

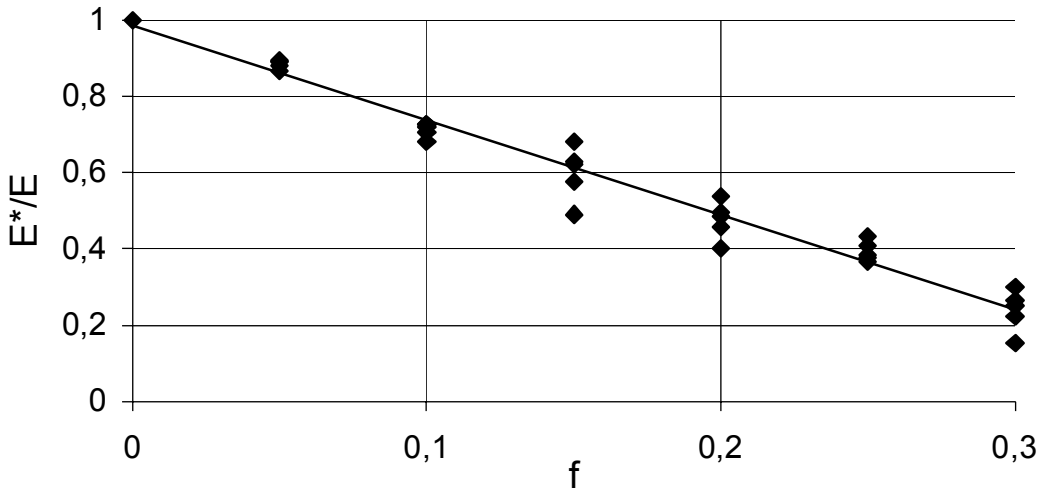

Figure 6: Influence of $f=n/N$ on the effective elastic modulus E^*.

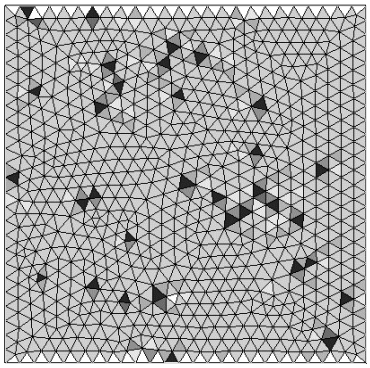

Figure 7: Stress concentrations near the void cells (black) in one of the simulations.

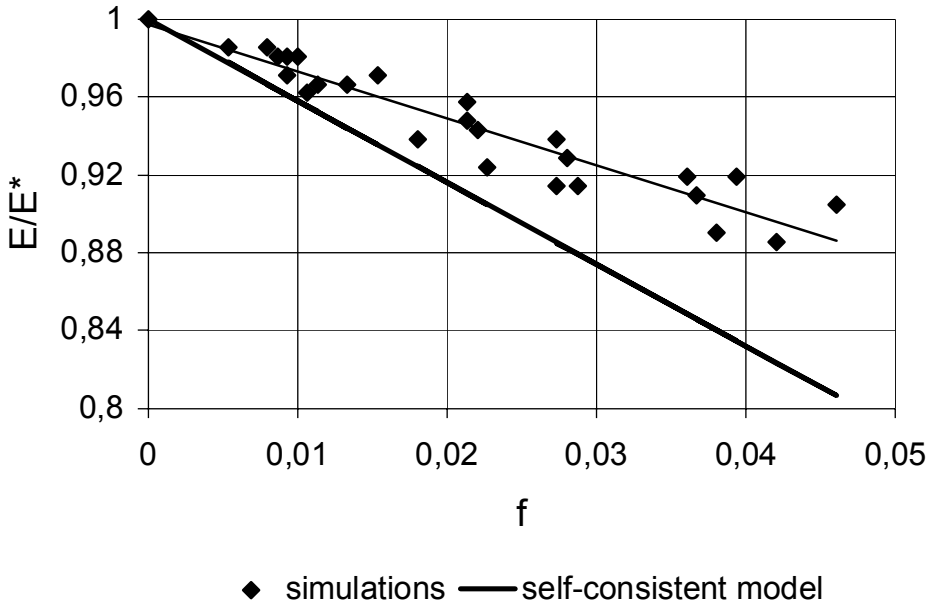

Figure 8: Low voids concentration: E^*/E vs. $f=n/N$.

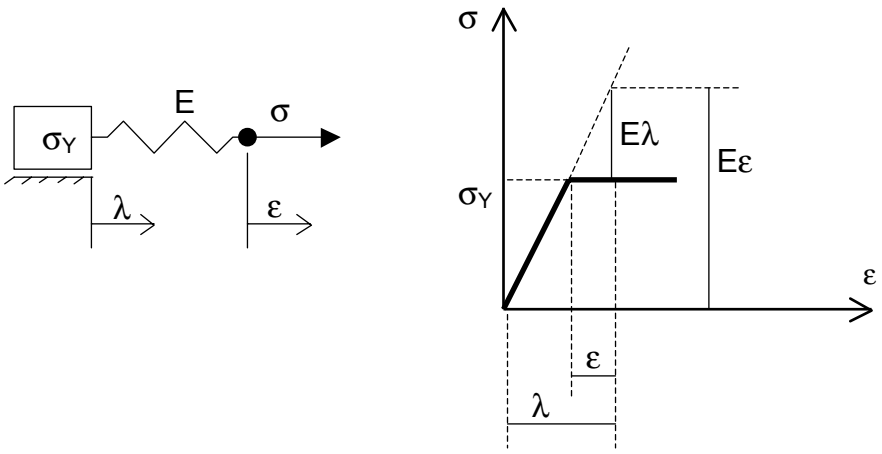

Figure 9: Mechanical model for an elastic-perfectly plastic material.

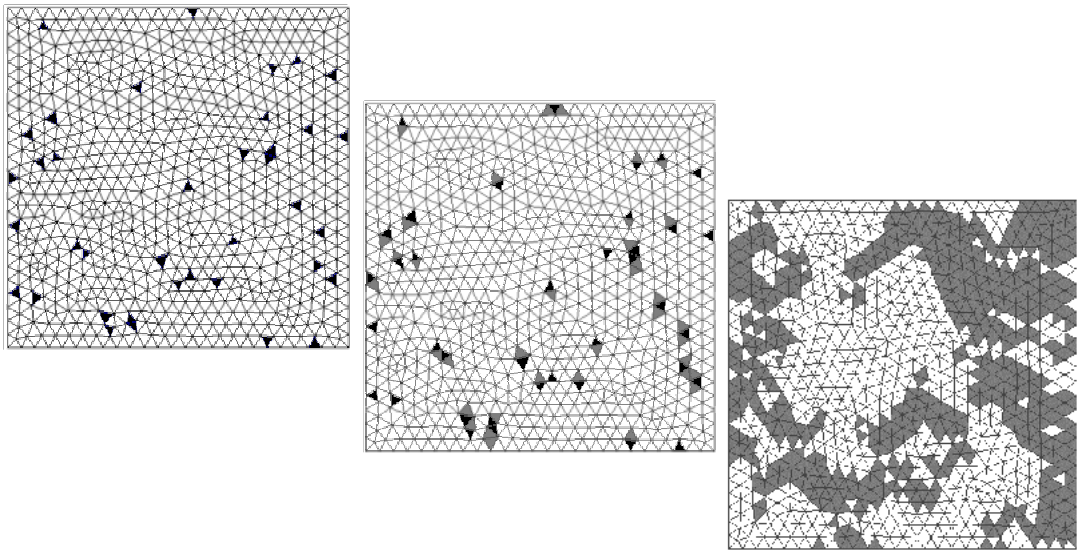

Figure 10: Progressive plasticization of the model. Void cells (black) among the material in elastic (white) and plastic (grey) field.

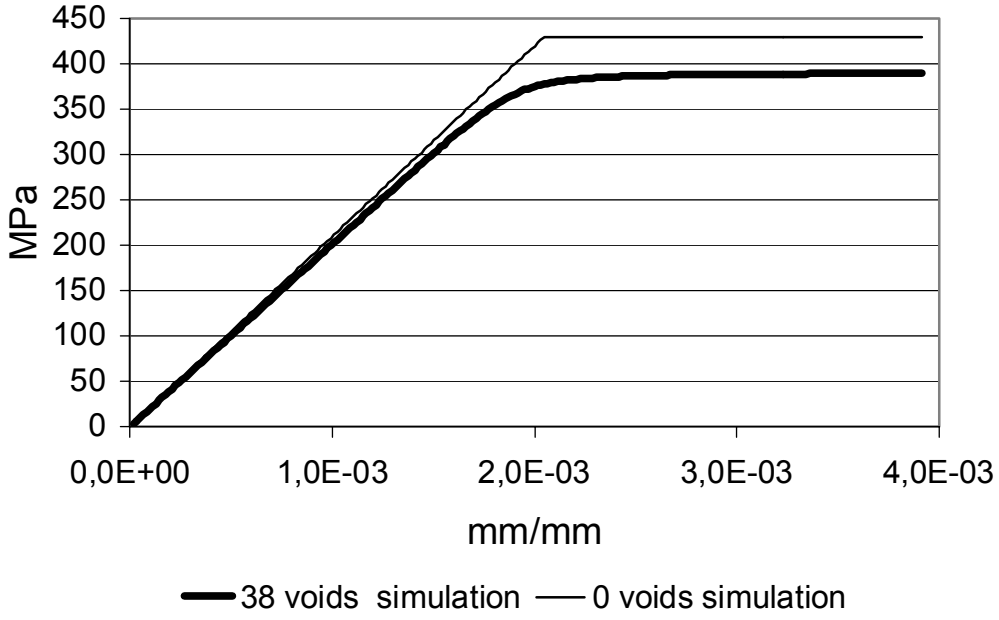

Figure 11: Stress-strain plot: 1498 primal cells, with 38 and without void cells.