The second order solution of Boussinesq's problem

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ABSTRACT: In this document, we have proposed a second order solution to Boussinesq's problem, which allows us to account for the experimental evidence on the stress field induced by aircraft traffic, showing that a tensile state of stress, not accounted for in the classical solution of Boussinesq's problem, always arises in the proximity of the contact area. The second order solution also allows us to evaluate the effect of the elastic constants on the stress field, improving the solution of Boussinesq in this second case also.

1 INTRODUCTION

The classical solution of Boussinesq (Boussinesq 1885) has been improved making use of derivatives of the second order (Ferretti 2012a). The second order solution for $r \neq 0$ is:

$$\begin{cases} \tau_{xz} = 3\dfrac{(x-x_1)z}{r^5}\left[z + 2C\mu\left(5\dfrac{z^2}{r^2} - \dfrac{\lambda}{\lambda+\mu}\right)\right]dm \\ \tau_{yz} = 3\dfrac{(y-y_1)z}{r^5}\left[z + 2C\mu\left(5\dfrac{z^2}{r^2} - \dfrac{\lambda}{\lambda+\mu}\right)\right]dm \\ \sigma_z = -\dfrac{1}{r^3}\left[3\dfrac{z^3}{r^2} + 2C\mu\left(15\dfrac{z^4}{r^4} - 3\dfrac{2\lambda+\mu}{\lambda+\mu}\dfrac{z^2}{r^2} - \right.\right. \\ \qquad \left.\left. -\dfrac{\lambda+2\mu}{\lambda+\mu}\right)\right]dm \end{cases} \tag{1}$$

where the terms in round brackets come from the second order solution. In particular, in the third of Equations 1, the new terms significantly modify the normal stress when approaching the surface, while they are negligible at great depths.

Since the normal stresses σ_z for $z \to 0$ and $z \to \infty$ are opposite in sign, near to the surface the compressed soil is subjected to a normal stress of traction.

2 NUMERICAL RESULTS

2.1 *Point-load perpendicular to the surface*

The plots of the vertical stress and the vertical stress contours of the second order solution for a prefixed Poisson's ratio, υ, and variable values of Young's modulus, E, are given in Figure 1 for a plane near to the surface and in Figure 2 for a vertical cross-section passing through the point load, respectively. The numerical solution of the second order shows two positive peaks of vertical stress in the proximity of the application point of compression load (Fig. 1), in total agreement with the experimental data for vehicular loading shown in Ferretti & Bignozzi (2012) and Ferretti (2012b). This result gives a numerical proof that a tensile state of stress arises on the surface of soils and pavements when subjected to compression loads, with the point in which the vertical stress change in sign that is also a point in which the vertical stress does not depend upon the value of E (Fig. 1). Since both soil and concrete are assumed as not being resistant to traction – to be on the safe side – the tensile state of stress must be considered with particular attention in these materials.

Due to the tensile state of stress, there exist two families of stress contours (Fig. 2): one family for the tensioned soil and one family for the compressed soil, with the two families separated by straight lines. Moreover, the parametric analysis on the vertical cross-section shows that greater values of E increase the vertical stresses at each depth without modifying the shape of the iso-lines of stress, which change in size homothetically (Fig. 2).

2.2 *Distributed load perpendicular to the surface*

The vertical stress and the vertical stress contours of the second order solution for a vertical uniform load over two contact areas of rectangular shape have been plotted in Figures 3, 4, respectively. The existence of a tensile state of stress in the proximity of the two contact areas is well evident, with the interaction effect making particularly severe the positive stresses between them.

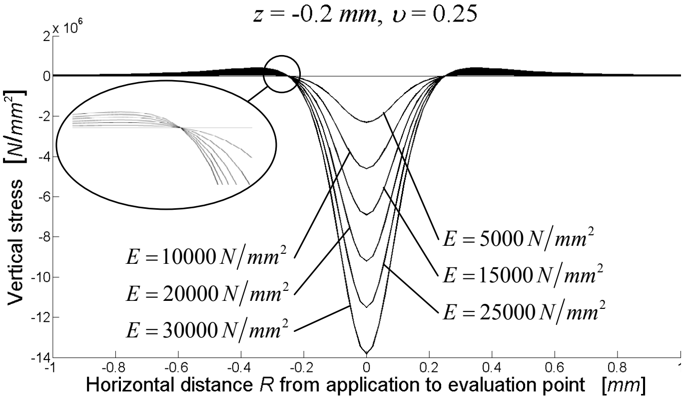

Figure 1. Parametric analysis on the Young modulus, *E*.

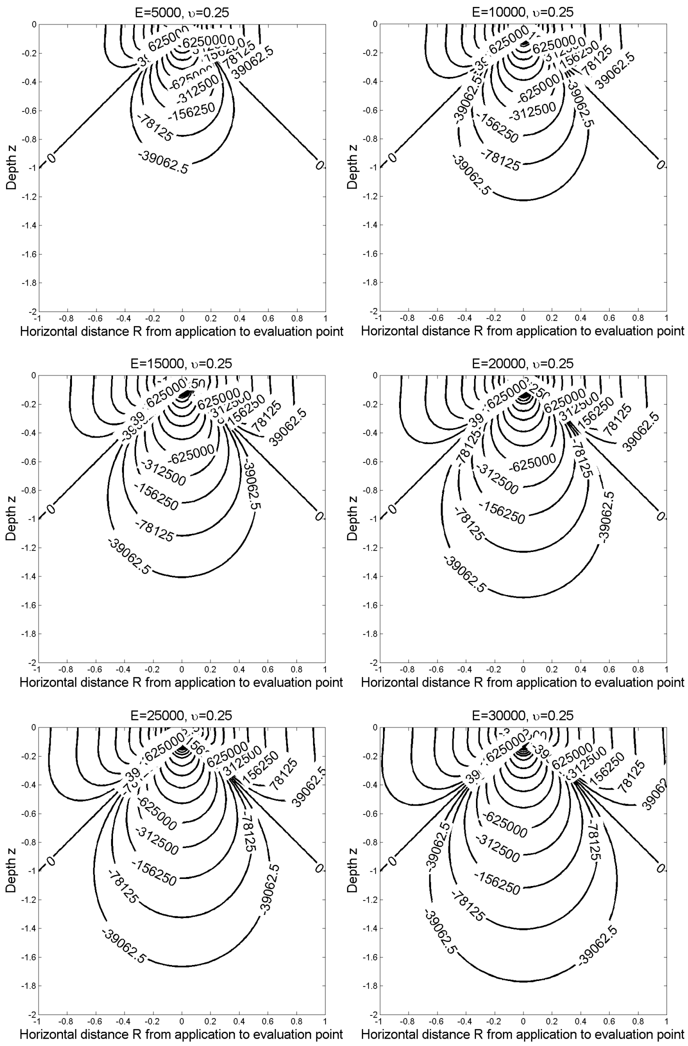

Figure 2. Parametric analysis on the Young modulus, *E*, for the vertical stress contours on the vertical cross-section (all distances in *mm*).

3 CONCLUSIONS

The second order solution of the equilibrium problem for a homogeneous linear-elastic and isotropic half-space allows us to evaluate the effect of the elastic constants on the stress field, which is an improvement to Boussinesq's solution.

Some numerical results have been provided, showing how the second order solution gives positive stresses at the surface, in the proximity of the compression load, both for point-load and distributed load. Further results for contact areas of circular, rectangular and elliptic shape together with uniform and parabolic laws of external pressure distribution may be found in Ferretti (in prep.).

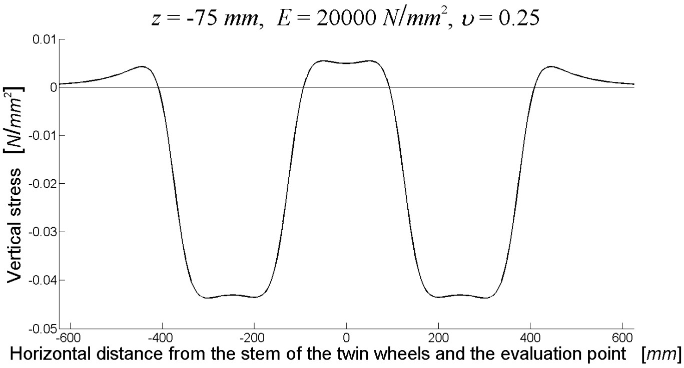

Figure 3. Vertical stress under the twin wheels of an aircraft for rectangular contact areas and uniform load.

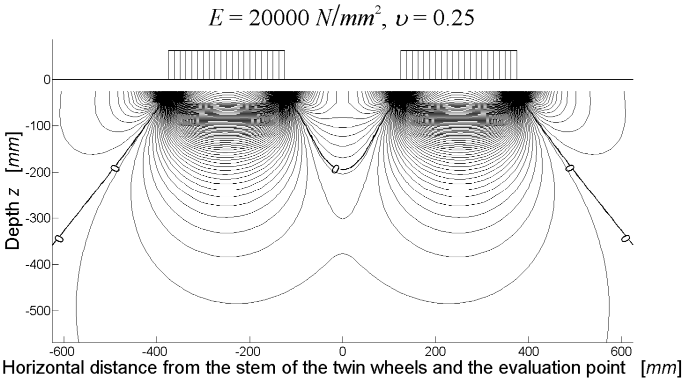

Figure 4. Vertical stress contours on the vertical cross-section for rectangular contact areas and uniform load.

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